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SRM Assignment 2

1. i) Generator matrix:

$$\begin{array}{c} B \\ D \end{array} \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

ii) The holding times are exponentially distributed with parameter λ in state B and μ in state D.

iii) $$\frac{\partial}{\partial t} {}_tP_s^{BB} = -\lambda \cdot {}_tP_s^{BB} + \mu \cdot {}_tP_s^{BD}$$

$$\frac{\partial}{\partial t} {}_tP_s^{BD} = \lambda \cdot {}_tP_s^{BB} - \mu \cdot {}_tP_s^{BD}$$

iv) We have 2 state model,

$${}_tP_s^{BB} + {}_tP_s^{BD} = 1$$

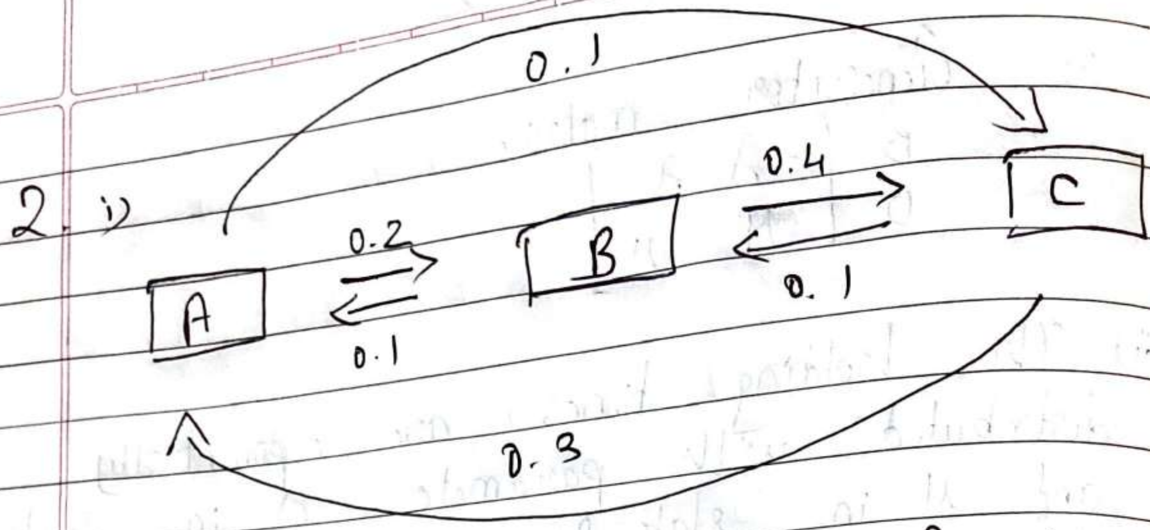
Substituting

$$\frac{\partial}{\partial t} {}_tP_s^{BB} = -\lambda \cdot {}_tP_s^{BB} + \mu (1 - {}_tP_s^{BB})$$

$$\therefore \frac{\partial}{\partial t} \left[e^{((\lambda+\mu)t)} \cdot {}_tP_s^{BB} \right] = \mu \cdot e^{((\lambda+\mu)t)}$$

$$\therefore e^{((\lambda+\mu)t)} \cdot {}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} \cdot e^{((\lambda+\mu)t)} + \text{constant}$$

2. i)



ii) $\frac{d}{dt} P_{AA}(t) = -0.3 P_{AA}(t) + 0.1 P_{AB}(t) + 0.3 P_{AC}(t)$

$\frac{d}{dt} P_{AB}(t) = 0.2 P_{AA}(t) - 0.5 P_{AB}(t) + 0.1 P_{AC}(t)$

$\frac{d}{dt} P_{AC}(t) = 0.1 P_{AA}(t) + 0.4 P_{AB}(t) - 0.4 P_{AC}(t)$

iii) To stay in state A the equation reduces,

$\frac{d}{dt} P_{\bar{A}\bar{A}}(t) = -0.3 P_{\bar{A}\bar{A}}(t)$

$P_{\bar{A}\bar{A}}(t) = e^{(-0.3t)}$

So for $t = 2$, $e^{(-0.6)} = 0.549$

iv) The only paths under which the third jump is into state C are BAC, CAC and CBC.
The probabilities of each jump are given by the ratio of the transition rates.

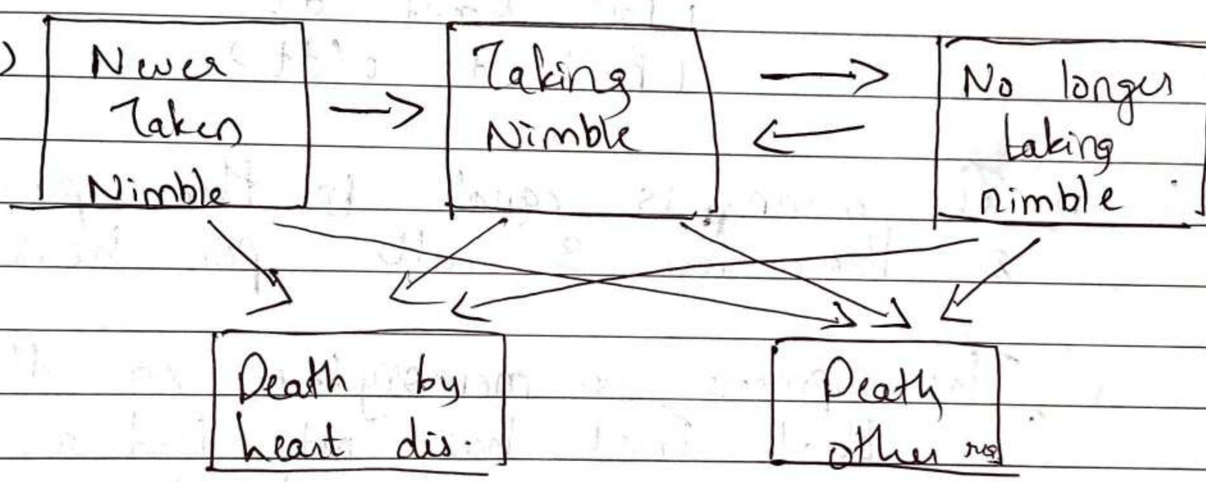
$BAC = \frac{2}{3} \cdot \frac{1}{5} \cdot \frac{1}{3} = \frac{2}{45}$

$CAC = \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{12}$

$CBC = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{4}{5} = \frac{1}{15}$

Sum = 0.1945

3. i)



ii) Chapman Kolmogrov equation is:

$$_{dt+t}P_x^{34} = {}_tP_x^{31}\,{}_{dt}P_{x+t}^{14} + {}_tP_x^{32}\,{}_{dt}P_{x+t}^{24} + {}_tP_x^{33}\,{}_{dt}P_{x+t}^{34} + {}_tP_x^{34}\,{}_{dt}P_{x+t}^{44} + {}_tP_x^{35}\,{}_{dt}P_{x+t}^{54}.$$

Since ${}_{dt}P_{x+t}^{54} = {}_tP_x^{31} = 0$.

$$_{dt+t}P_x^{34} = {}_tP_x^{32}\,{}_{dt}P_{x+t}^{24} + {}_tP_x^{33}\,{}_{dt}P_{x+t}^{34} + {}_tP_x^{34}\,{}_{dt}P_{x+t}^{44}$$

Given, ${}_{dt}p_{x+t}^{44} = 1$

For small dt,

$${}_{dt}p_{x+t}^{ij} = \mu_{x+t}^{ij}\, dt + o(dt) \quad i \neq j$$

where $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$,

$${}_{dt+t}p_x^{34} = {}_tp_x^{32}\, \mu_{x+t}^{24}\, dt + {}_tp_x^{33}\, \mu_{x+t}^{34}\, dt + {}_tp_x^{34} + o(dt)$$

4\. i) The mean is equal to the parameter so there are 3 calls per hour.

ii) The process is memoryless so that fact that Fred has not had a call for 15 minutes is irrelevant.

iii) This is the probability of 0 calls in time 0.5 hours.

Using $p_j(t) = e^{-\lambda t}(\lambda t)^j / j!$.

$$p_0(0.5) = 0.222$$

iv) The expected time that Fred is on the phone is the expected number of calls times the expected length of a call.

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Per hour this is 3 calls times 7 minutes = 21 minutes
Hence probability that phone is engaged is $\frac{21}{60} = 0.35$ //

5\. Chapman Kolmogrov equation is,

$$P_{HH}(x, t+dt) = P_{HH}(x, t)\, P_{HH}(t, t+dt) + P_{HS}(x, t)\, P_{SH}(t, t+dt) + P_{HD}(x, t)\, P_{DH}(t, t+dt)$$

But $P_{DH}(t, t+dt) = 0$ or other explanation why path through D can be ignored

$$P_{HH}(x, t+dt) = P_{HH}(x, t)\, P_{HH}(t, t+dt) + P_{HS}(x, t)\, P_{SH}(t, t+dt)$$

Assuming for small dt.

$$p_{ij}(t, t+dt) = \lambda_{ij}(t)\, dt + o(dt)$$

$$p_{ii}(t, t+dt) = 1 + \lambda_{ii}(t)\, dt + o(dt)$$

Substituting

$$P_{HH}(x, t+dt) = P_{HH}(x, t)(1 - \sigma(t)\, dt - \mu(t)\, dt) + P_{HS}(x, t)\, \rho(t, C_t) + o(dt)$$

$$\therefore \frac{d}{dt} P_{HH}(x, t) = \lim_{dt \to 0} \frac{P_{HH}(x, t+dt) - P_{HH}(x, t)}{dt}$$

11. i) Let N_t denote the number of claims up to time t. Since the poisson process has stationary increments, we may take $t=0$ so that the required conditional distribution is.

$$P(T_0 \le y \mid N_s = 1) = \frac{P(T_0 \le y \quad N_s = 1)}{P(N_s = 1)}$$

$$= \frac{P(N_y = 1 \quad N_s - N_y = 0)}{P(N_s = 1)}$$

But $N_s - N_y$ is independent of N_y

Thus, $$\frac{(\lambda y e^{-\lambda y}) \cdot e^{-\lambda(s-y)}}{\lambda s e^{-\lambda s}} = \frac{y}{s}$$

which is the cdf of uniform dist on $[0, s]$

ii) Since holding times are independent each having an exponential distribution, their joint density is

$$\lambda^n e^{-\lambda(t_1 + t_2 + \dots + t_n)} 1\{t_1, t_2 \dots t_n > 0\}$$

ii) Since we have,

$$P(N_s = k \mid N_t = n)$$
$$= \frac{P(N_s = k, N_t = n)}{P(N_t = n)}$$

$$= \frac{P(N_s = k, N_t - N_s = n-k)}{P(N_t = n)}$$

Using again the Poisson process has stationary and independant increments and that the number of claim in an interval $[0, t]$ is Poisson (t),

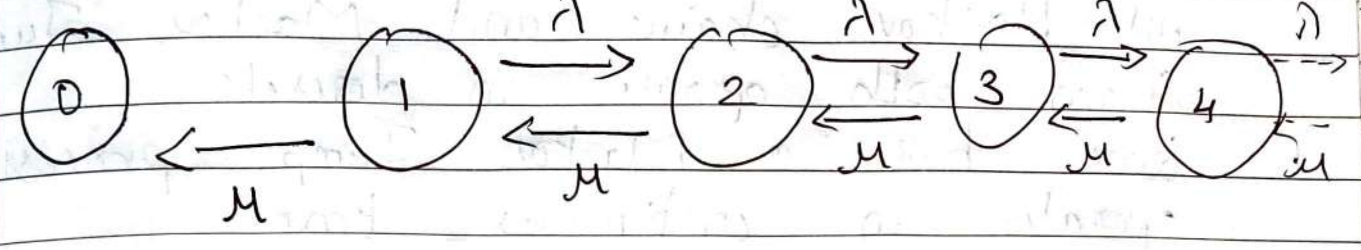
$$P(N_s = k \mid N_t = n)$$
$$= \frac{\frac{e^{-\lambda s}(\lambda s)^k}{k!} \cdot \frac{e^{-\lambda(t-s)}\lambda^{n-k}(t-s)^{n-k}}{(n-k)!}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{e^{-\lambda t}\lambda^n s^k (t-s)^{n-k}}{k!\,(n-k)!} \times \frac{n!}{e^{-\lambda t}\lambda^n t^n}$$

$$= \binom{n}{k}\left(\frac{s}{t}\right)^k\left(1-\frac{s}{t}\right)^{n-k}.$$

8- i) $\{0, 1, 2, 3, 4 \ldots\}$

ii)



iii) Generator matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 \\ 0 & \mu & -(\mu+\lambda) & \lambda & 0 \\ 0 & 0 & \mu & -(\mu+\lambda) & \lambda \\ 0 & 0 & 0 & \mu & -(\mu+\lambda \end{bmatrix}$$

iv) If a Markov jump process X_t is examined only at times of transition the resulting process is called jump chain associated with X_t.

vi) $\left(\frac{\mu}{\mu+\lambda}\right)^3$.

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6. All three processes have a discrete state space.

All Markov chain and Markov Jump Chain both operate in discrete time but a Markov Jump process operates in continuous time.

All have the Markov property is that the future development of the process can be predicted from its present state alone, without reference to its past history.

If a Markov Jump Process X is examined only at the times of its transitions, the resulting process is called the Jump Chain associated with X.

The Jump Chain and the Markov Chain are expressed in terms of probabilities whereas the Markov Jump Process is expressed in terms of rates.

The Markov Chain can have loops in each state, the Markov Jump process cannot and the Markov Jump Chain only has loops on absorbing states.

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7\. i) Entry and exit time for each individual for each state and the total no. of transactions of each type made.

ii) Define $P_{AA}(s, t)$ to be the probability of being in state Active at time s+t if Active at time s.

$$\frac{\partial}{\partial t} P_{AA}(s, t) = -P_{AA}(s, t)\mu$$

$$\frac{\partial}{\partial t} P_{AI}(s, t) = P_{AA}(s, t)\mu$$

iii) Measure from time zero i.e s = 0 and drop s from notation.

$$\frac{1}{P_{AA}(t)} \frac{\partial}{\partial t} P_{AA}(t) = -\mu$$

$$\frac{\partial}{\partial t} (\ln (P_{AA}(t))) = -\mu$$

$$\therefore P_{AA}(t) = e^{(-\mu t + C)}$$

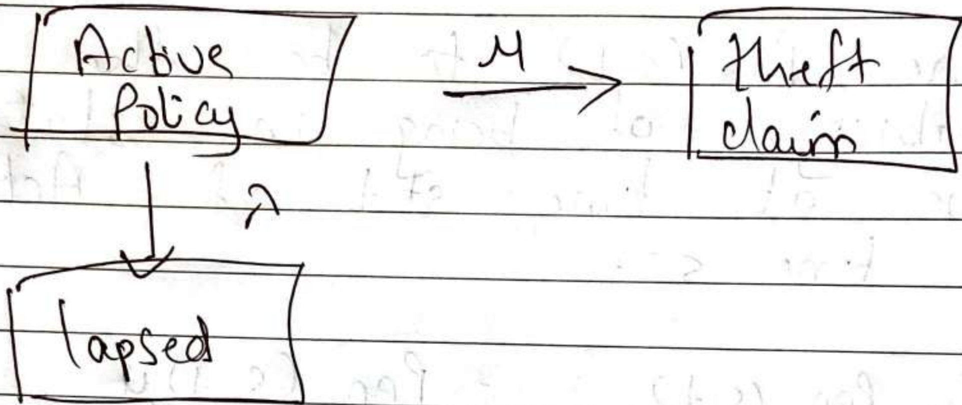
$$\therefore P_{AA}(0) = 1 \quad C = 0 \quad \text{so}$$

$$P_{AA}(t) = e^{(-\mu t)}$$

A claim occurs with cost £C if moves to state "Theft claim".

Hence the expected cost is $C(1-e^{(-\mu T)})$

iv)



10. i) $\frac{d}{dt} P_{\bar{A}\bar{A}}(t) = -2t \times P_{\bar{A}\bar{A}}(t)$

$\Rightarrow \frac{d}{dt}\left[\ln P_{\bar{A}\bar{A}}(t)\right] = -2t$

$\ln P_{\bar{A}\bar{A}}(s) = -s^2 + \text{constant}$

We know $P_{\bar{A}\bar{A}}(0) = 1$, constant $= 0$.

Hence $P_{\bar{A}\bar{A}}(s) = e^{-s^2}$

ii) P(in first visit to B at time T in state A at t = 0)

$= \int_0^T$ P(remains in A to time s) $\times$ P(transition to B in time s, s+ds) $\times$ P(remain in B to time T) ds

$= \int_{s=0}^{T} P_{\bar{A}\bar{A}}(s) \times 2s \times P_{\bar{B}\bar{B}}(s, T)\, ds$

Using the result from part (i) and similarly for P_{BB} with boundary condition $P_{BB}(s, s) = 1$

$= \int_{s=0}^{T} e^{-s^2} \times 2s \times e^{-T^2+s^2}\, ds$

$= e^{-T^2} \times T^2$