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Assignment - II

(1) X = claim size on a home insurance

$X \sim \text{Normal}(800, 100^2)$

Y: claim size on a car insurance

$Y \sim \text{Normal}(1200, 300^2)$

$P(Y_1+Y_2+Y_3 > X_1+X_2+X_3+X_4 + 800)$

$= P((Y_1+Y_2+Y_3) - (X_1+X_2+X_3+X_4) > 800)$

$(Y_1+Y_2+Y_3) - (X_1+X_2+X_3+X_4) \sim N(3(1200) - 4(800), (3\times 300^2 + 4\times 100^2))$

$\sim N(400, 310000)$

$P((Y_1+Y_2+Y_3) - (X_1+X_2+X_3+X_4) > 800)$

$= P\left(Z > \frac{800-400}{\sqrt{310000}}\right) = P(Z > 0.718)$

$= 1 - P(Z < 0.718)$

$\approx 1 - 0.76424$

$\simeq 0.23576$

(2) N = number of claims

$N \sim \text{Negative Binomial}(3, 0.9)$

X = Claim amount

$X \sim \text{gamma}(6, 2)$

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Y = Total claim amount

$\cancel{M_Y(t) = E(e^{tY}}$

$$M_Y(t) = M_N(\log M_X(t))$$

$$M_X(t) = \left(1 - \frac{t}{2}\right)^{-6}$$

$$M_Y(t) = M_N\left(\log\left(1 - \frac{t}{2}\right)^{-6}\right)$$

$$\ldots \left(M_N(t) = \left(\frac{pe^t}{1-qe^t}\right)^k\right)$$

$$M_Y(t) = \left(\frac{pe^{\log(1-\frac{t}{2})^{-6}}}{1 - qe^{\log(1-\frac{t}{2})^{-6}}}\right)^3$$

$$M_Y(t) = \left(\frac{p\left(1-\frac{t}{2}\right)^{-6}}{1 - q\left(1-\frac{t}{2}\right)^{-6}}\right)^3$$

$$M_Y(t) = \left(\frac{0.9\left(1-\frac{t}{2}\right)^{-6}}{1 - (0.1)\left(1-\frac{t}{2}\right)^{-6}}\right)^3$$

$E(N) = \frac{3}{0.9} = \frac{10}{3}$

$V(N) = \frac{k(0.1)}{(0.9)^2} = \frac{3}{0.9}\left(\frac{1}{9}\right) = \frac{10}{27}$

$E(X) = \frac{\alpha}{\lambda} = \frac{6}{2} = 3$ $\quad$ $V(X) = \frac{\alpha}{\lambda^2} = \frac{6}{4} = \frac{3}{2}$

$V(Y) = E(N)V(X) + E(X)^2 V(N)$

$= \frac{\cancel{10}^{5}}{\cancel{3}}\left(\frac{\cancel{3}}{\cancel{2}}\right) + \cancel{9}\left(\frac{10}{\cancel{27}_{3}}\right)$

$= 5 + \frac{10}{3} = \frac{25}{3}$

$S.D(Y) = \sqrt{\frac{25}{3}} = \cancel{2.0274}\ \underline{2.886}$

③ $f_X(x) = \lambda e^{-\lambda(x-\alpha)}$

$M_X(t) = E(e^{tX})$

$= \int_\alpha^\infty e^{tx} \lambda e^{-\lambda(x-\alpha)} dx$

$= \lambda \int_\alpha^\infty e^{tx} e^{-\lambda x} e^{\lambda\alpha} dx$

$= \lambda e^{\lambda\alpha} \int_{\alpha}^\infty e^{x(t-\lambda)} dx$

$= \frac{\lambda e^{\lambda\alpha}}{t-\lambda}\left[e^{(t-\cancel{\lambda})x}\right]_0^\infty = \frac{\lambda e^{\lambda\alpha}}{t-\lambda}\left[e^{-(\lambda-t)x}\right]_0^\infty$

$$M_x(t) = \frac{\lambda e^{\lambda\alpha}}{t-\lambda}\left[0 - e^{-\lambda\alpha}\cdot e^{t\alpha}\right]$$

$$\boxed{M_x(t) = \frac{\lambda e^{t\alpha}}{\lambda - t}}$$

~~ii.~~ $\cancel{M'_x(t) = \frac{\lambda}{(\lambda-t)^2}}$

ii. $$M'_x(t) = \lambda(e^{t\alpha})(\lambda-t)^{-2}(-1)(-1) + \lambda(\lambda-t)^{-1}(e^{t\alpha})(\alpha)$$

$$= \frac{\lambda e^{t\alpha}}{(\lambda-t)^{+2}} + \frac{\lambda e^{t\alpha}\alpha}{\lambda - t}$$

$$E(x) = M'_x(0) = \frac{\lambda(1)}{\lambda^2} + \frac{\lambda\alpha}{\lambda}$$

$$= \frac{1+\lambda\alpha}{\lambda} = \frac{1+\lambda\alpha}{\lambda}$$

$$M''_x(t) = \lambda e^{t\alpha}(-2)(\lambda-t)^{-3}(-1) + \lambda(\lambda-t)^{-2}(e^{t\alpha})(\alpha)$$
$$+ \lambda\alpha e^{t\alpha}(\lambda-t)^{-2}(-1)(-1)$$
$$+ \lambda\alpha(\lambda-t)^{-1}(e^{t\alpha})(\alpha)$$

$$= \frac{2\lambda e^{t\alpha}}{(\lambda-t)^3} + \frac{\lambda\alpha e^{t\alpha}}{(\lambda-t)^2} + \frac{\lambda\alpha e^{t\alpha}}{(\lambda-t)^2} + \frac{\alpha^2\lambda e^{t\alpha}}{\lambda - t}$$

$$= \frac{2\lambda e^{t\alpha}}{(\lambda-t)^3} + \frac{2\lambda\alpha e^{t\alpha}}{(\lambda-t)^2} + \frac{\lambda\alpha^2 e^{t\alpha}}{\lambda - t}$$

$$E(x^2) = M''_x(0) = \frac{2\cancel{\lambda}(1)}{\lambda^{3}} + \frac{2\cancel{\lambda}\alpha}{\lambda^{2}} + \frac{\cancel{\lambda}\alpha^2}{\cancel{\lambda}}$$

$$= \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= \alpha^2 + \frac{2\alpha}{\lambda} + \frac{2}{\lambda^2} - \left(\frac{1 + \lambda^2\alpha^2 + 2\alpha\lambda}{\lambda^2}\right)$$

$$= \cancel{\alpha^2} + \cancel{\frac{2\alpha}{\lambda}} + \frac{2}{\lambda^2} - \frac{1}{\lambda^2} - \cancel{\alpha^2} - \cancel{\frac{2\alpha}{\lambda}}$$

$$\boxed{V(x) = \frac{1}{\lambda^2}}$$

(4) $G_V(t) = \left(\frac{p}{1-qt}\right)^k, \quad p+q=1$

$\therefore$ given distribution is negative binomial type 2

$$P(V=4) = \frac{\Gamma(k+4)}{\Gamma 5 \, \Gamma k} p^4 q^4$$

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(5) $f(x,y) = ax^2 + axy$ $\qquad 0 < x < 5$

$10 < y < 20$

$$\Rightarrow 1 = \int_{10}^{20} \int_0^5 ax^2 + axy \, dx \, dy$$

$$1 = \int_{10}^{20} \left[\frac{ax^3}{3}\right]_0^5 + \left[ay\frac{x^2}{2}\right]_0^5 dy$$

$$1 = \int_{10}^{20} \frac{125a}{3} + \frac{25}{2}ay \, dy$$

$$1 = \frac{125a}{3}\left[y\right]_{10}^{20} + \frac{25a}{2}\left[\frac{y^2}{2}\right]_{10}^{20}$$

$$1 = \frac{1250a}{3} + 1875a$$

$$\boxed{a = \frac{3}{6875}}$$

~~$E(Y|X) = y f_{Y|X}(y|x$~~ $\qquad f_X(x) = \int_{10}^{20} a(x^2 + xy)a \, dy$

~~$E(Y|X) =$~~ $\qquad = \frac{3}{6875}\int_{10}^{20} (x^2 + xy) \, dy$

$$= \frac{3}{6875}\left[x^2\left[y\right]_{10}^{20} + x\left[\frac{y^2}{2}\right]_{10}^{20}\right]$$

$$= \frac{3}{6875}\left(10x^2 + 150x\right)$$

$$E(Y|X) = y f(Y|X) = y \int_{10}^{20} \frac{f(x,y)}{f_X(x)}$$

$$= \int_{10}^{20} \frac{y\left(\frac{\cancel{3}}{6875}\right)(x^2 + xy)}{\frac{\cancel{3}}{6875}(10x^2 + 150x)} \, dy$$

$$= \frac{1}{10} \int_{10}^{20} \frac{y(x+y)}{(x+15)} \, dy$$

$$= \frac{1}{10(x+15)} \left[\left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_{10}^{20} \right]$$

$$= \frac{1}{10(x+15)} \left(150x + \frac{7000}{3} \right)$$

$$= \frac{1}{x+15} \left(15x + \frac{700}{3} \right) \Rightarrow \frac{45x + 700}{3(x+15)}$$

$$\cancel{= \frac{745}{3(x+15)}}$$

$$\boxed{E(Y|X) = \frac{45x + 700}{3(x+15)}}$$

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$P\left(Y > \frac{1}{3}X + 10\right) \Rightarrow$

$$f_Y(y) = \int_0^5 \frac{3}{6875}(x^2 + xy)\,dx$$

$$= \frac{3}{6875}\left[\frac{x^3}{3} + \frac{x^2}{2}y\right]_0^5$$

$$= \frac{3}{6875}\left[\frac{125}{3} + \frac{25y}{2}\right]$$

$$= \frac{1}{6875}\left[\frac{250 + 75y}{2}\right]$$

$$P\left(Y > \frac{1}{3}X + 10\right) = \int_{\frac{1}{3}X+10}^{20} \frac{1}{6875}\frac{(250 + 75y)}{2}\,dy$$

$$= \frac{1}{6875(2)}\left[250\left[y\right]_{\frac{1}{3}X+10}^{20} + 75\left[\frac{y^2}{2}\right]_{\frac{X}{3}+10}^{20}\right]$$

$$= \frac{1}{6875(2)}\left\{(250)\left(20 - \frac{X}{3} - 10\right) + 75\left(200 - \frac{X^2}{9} - 100 - \frac{20X}{3}\right)\right\}$$

$$= \frac{1}{6875(2)}\left[5000 - \frac{250X}{3} - 2500 + 15000 - \frac{75X^2}{3} - 7500 - \frac{1500X}{3}\right]$$

$$= \frac{1}{6875(2)}\left[10000 - \frac{1750X}{3} - \frac{25X^2}{3}\right]$$

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$X \sim Poi(\lambda) \qquad Y \sim Poi(\mu)$

$M_X(t) = e^{\lambda(e^t - 1)} \qquad M_Y(t) = e^{\mu(e^t - 1)}$

$$M_{X-Y}(t) = E\left(e^{t(X-Y)}\right)$$
$$= E\left(e^{Xt - Yt}\right)$$
$$= \frac{E(e^{Xt})}{E(e^{Yt})} = \frac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}} = e^{(\lambda-\mu)(e^t-1)}$$

$\therefore \quad X - Y \sim Poi(\lambda - \mu)$

$\therefore$ Hence Proved

$$M_{X+Y}(t) = E\left(e^{t(X+Y)}\right) = E\left(e^{tX + Yt}\right)$$
$$= E(e^{tX}) \cdot E(e^{tY})$$
$$= e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)}$$
$$= e^{(\lambda+\mu)(e^t-1)}$$

$\therefore \quad X + Y \sim Poi(\lambda + \mu)$

$\therefore$ Hence Proved

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$$Var(X|Y=2) = E(X^2|Y=2) - (E(X|Y=2))^2$$
$$= \frac{\sum_x x^2 P(X=x, Y=2)}{P(Y=2)} - \left(\frac{\sum_x x P(X=x, Y=2)}{P(Y=2)}\right)^2$$

$$= \frac{1^2(0) + 2^2(0.3) + 3^2(0.1) + 4^2(0.\overset{2}{\cancel{15}})}{0.6} - \left[\frac{1(0) + 2(0.3) + 3(0.1) + 4(0.\overset{2}{\cancel{15}})}{0.6}\right]^2$$

$\cancel{= 7.5 - 6.25 = 1.25}$ $= \frac{53}{6} - \frac{289}{36} = \frac{29}{36}$

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ii. $f(u,v) = \frac{48}{67}(2uv - u^2)$, $0 < u < 1$, $\frac{u}{2} < v < 2$

$$E(U|V=v) = \int_0^1 u\, f(u|V=v)\, du$$

$$= \int_0^1 u \frac{f(u, V=v)}{f(V=v)}\, du \quad -①$$

$$\Rightarrow f_V(V=v) = \int_0^1 \frac{48}{67}(2uv - u^2)\, du$$

$$= \frac{48}{67}\left[\frac{\cancel{2}u^2 v}{\cancel{2}} - \frac{u^3}{3}\right]_0^1$$

$$= \frac{48}{67}\left[v - \frac{1}{3}\right]$$

$$= \frac{48[3v-1]}{67(3)} = \frac{16(3v-1)}{67}$$

$$E(U|V=v) = \int_0^1 \frac{u\left(\frac{\cancel{48}^{3}}{\cancel{67}}\right)(2uv - u^2)}{\frac{\cancel{16}}{\cancel{67}}(3v-1)}\, du$$

$$= \frac{3}{(3v-1)}\int_0^1 (2u^2 v - u^3)\, du$$

$$= \frac{3}{(3v-1)}\left[\frac{2u^3 v}{3} - \frac{u^4}{4}\right]_0^1 = \frac{3}{(3v-1)}\left(\frac{2v}{3} - \frac{1}{4}\right)$$

$$\boxed{E(U|V=v) = \frac{(8v-3)}{4(3v-1)}}$$

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(8) $X \sim \text{Gamma}(3,2)$

$E(Y|X=x) = 3x+1$ $\quad$ $Var(Y|X=x) = 2x^2+5$

$E(E(Y|X=x)) = E(Y)$

$E(Y) = E(3x+1)$

$= 3E(x)+1$

$= 3\left(\frac{3}{2}\right)+1$

$= \frac{11}{2}$

$Var(Var(Y|X=x)) = V(Y)$

$V(Y) = E(Var(Y|X=x)) + V(E(Y|X=x))$

$= E(2x^2+5) + V(3x+1)$

$= 2E(x^2)+5+9V(x)$

$= 2(3)+5+9\left(\frac{3}{4}\right)$

$= 6+5+6.75$

$= \underline{\underline{17.75}}$

(9) $Z = X+Y$

$E(x)=3$ $\quad$ $E(y)=4$

$\sqrt{V(X)} = 2$ $\quad$ $\sqrt{V(Y)} = 1$

i. $Cov(X,Z) = Cov(X,X+Y)$ $\quad$ $Cor(X,Y) = -0.3$

$= Cov(X,X) + Cov(X,Y)$

$= V(X) + (-0.6)$

$= 4 - 0.6 = \underline{3.4}$

ii. $Var(Z) = Var(X+Y)$

$= V(X) + V(X) + 2Cov(X,Y)$

$= 4+1+2(-0.6)$

$= 5-1.2$

$= \underline{\underline{3.8}}$

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(10) $f(x,y) = kx^{-\alpha} e^{-y/\beta}$, $1 < x < \infty$
$1 < y < \infty$
$\alpha > 1$, $\beta > 0$

$$1 = k \int_1^\infty x^{-\alpha}\, dx \int_1^\infty e^{-y/\beta}\, dy$$

$$1 = k \left[\frac{x^{-\alpha+1}}{-\alpha+1} \right]_1^\infty \left[\frac{e^{-y/\beta}}{-1/\beta} \right]_1^\infty$$

$$1 = k \left[0 - \frac{1}{-\alpha+1} \right] \left[0 + \beta(e^{-1/\beta}) \right]$$

$$1 = k \left[\frac{1}{-\alpha+1} \right] \left[-\beta e^{-1/\beta} \right]$$

$$k = \frac{-1+\alpha}{\beta e^{-1/\beta}} = \frac{\alpha - 1}{\beta e^{-1/\beta}} = k$$