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Subject - Probability & Statistics 1
Chapter - Unit 3 and 4
Category - Assignment 2

Q1) Let H denoted home insurance policies and C denote car insurance policies.

$\therefore$ given : $H \sim N(800, 100^2)$

$C \sim N(1200, 300^2)$

All claims are independent

H amount to 800 but no car claim

$$P(H_1 + H_2 + H_3 + H_4 + 800 < C_1 + C_2 + C_3) = ?$$

$$\therefore P[(H_1 + H_2 + H_3 + H_4) - (C_1 + C_2 + C_3) < (-800)]$$

$$= P[(C_1 + C_2 + C_3) - (H_1 + H_2 + H_3 + H_4) > (800)]$$

Let $X = (C_1 + C_2 + C_3) - (H_1 + H_2 + H_3 + H_4)$

$C_1 + C_2 + C_3 \sim N(1200 \times 3, 300^2 \times 3)$

$H_1 + H_2 + H_3 + H_4 \sim N(800 \times 4, 100^2 \times 4)$

$$\therefore (C_1 + C_2 + C_3) - (H_1 + H_2 + H_3 + H_4) \sim$$

$$N(1200 \times 3 - 800 \times 4, 300^2 \times 3 + 4 \times 100^2)$$

$$\therefore X \sim N(400, 556.77644^2)$$

$$P(X > 800) = P\left(Z > \frac{800 - 400}{556.7764363}\right)$$

$$= P(Z > 0.7184)$$

$$= 1 - P(Z \leq 0.7184)$$

By linear Interpolation,

~~f(x~~ $x_1 = 0.71$ $\qquad f(x_1) = 0.76115$

$x_2 = 0.72$ $\qquad f(x_2) = 0.76424$

$x = 0.7184$ $\qquad f(x) = ?$

~~f(x)~~ By formula,

$$\frac{x - x_1}{x_2 - x_1} = \frac{f(x) - f(x_1)}{f(x_2) - f(x_1)}$$

$$\therefore f(x) = \frac{(x - x_1)(f(x_2) - f(x_1))}{x_2 - x_1} + f(x_1)$$

$$= \left[\frac{(0.7184 - 0.71)(0.76424 - 0.76115)}{0.72 - 0.71}\right] + 0.76115$$

$$= 0.7637456$$

$$\therefore P(X > 800) = 1 - 0.7637456$$

$$= 0.2362544$$

Q2) N is number of claims and it follows negative binomial distribution

$\therefore N \sim NBin(k = n+1, p) = (3, 0.9)$

[On comparing with :
given : $P(N=n) = {}^{n+2}C_n \; 0.9^3 \; 0.1^n$]

X be the total claim amounts

$X \sim Gamma(6, 2)$

$Y = X_1 + X_2 + X_3 \cdots X_N$

From Tables,

$\cancel{X_N}$ $M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-\alpha}$

$M_N(t) = \left(\frac{pe^t}{1-(1-p)e^t}\right)^k$

$\therefore M_X(t) = \left(1 - \frac{t}{2}\right)^{-6}$

$\ln M_X(t) = -6 \ln\left(1 - \frac{t}{2}\right)$

~~$M_N(t) =$~~

$M_Y(t) = M_N[\log M_X(t)]$

$= M_N\left[-6 \ln\left(1 - \frac{t}{2}\right)\right]$

$= M_N\left[\ln\left(1 - \frac{t}{2}\right)^{-6}\right]$

$$= \frac{p e^{\ln(1-\frac{t}{2})^{-6}}}{1-(1-p)\, e^{\ln(1-\frac{t}{2})^{-6}}}$$

$$M_Y(t) = \frac{p\left(1-\frac{t}{2}\right)^{-6}}{1-(1-p)\left(1-\frac{t}{2}\right)^{-6}}$$

(ii) $s.d = \sqrt{Var(Y)}$

$$= \sqrt{E(N)\, var(X) + var(N)\, [E(X)]^2}$$

from given information and tables:

$E(N) = \frac{n+1}{p} = \frac{3}{0.9}$ $\quad Var(N) = \frac{(n+1)(1-p)}{p^2} = \frac{3\times 0.1}{0.9^2}$

$E(X) = \frac{6}{2} = 3 \qquad V(X) = \frac{6}{2^2} = \frac{6}{4} = 1.5$

$\therefore \quad s.d = \sqrt{3.333 \times 1.5 + 0.37037 \times 3^2}$

$= 2.886751$

Q3 $f(x) = \lambda e^{-\lambda(x-\alpha)}$ $\quad x \geq \alpha$ & $\lambda; \alpha > 0$

$$MGF = M_x(t) = E(e^{tx})$$

$$= \int_0^{\infty} e^{tx} \, f(x) \, dx$$

$$= \int_0^{\infty} e^{tx} \, \lambda e^{-\lambda(x-\alpha)} \, dx$$

$$= \lambda \int_0^{\infty} e^{tx - \lambda x + \lambda \alpha} \, dx$$

$$= \lambda \cdot e^{\lambda \alpha} \int_0^{\infty} e^{tx - \lambda x} \, dx$$

$$= \lambda \cdot e^{\lambda \alpha} \left[\frac{e^{tx - \lambda x}}{t - \lambda} \right]_0^{\infty}$$

$$= \frac{-\lambda e^{\lambda \alpha}}{t - \lambda}$$

$$= \frac{\lambda e^{\lambda \alpha}}{\lambda - t}$$

$$= \frac{e^{\lambda \alpha}}{\left(\frac{\lambda - t}{\lambda} \right)}$$

$$= \cancel{e^{\lambda \alpha} \times \left(1 - \frac{t}{\lambda} \right)^{-1}}$$

$$= \cancel{\frac{1}{\frac{\lambda}{e^{\lambda\alpha} \cdot \lambda} \cdot \frac{t}{e^{\lambda\alpha} \cdot \lambda}}}$$

$$= e^{\lambda \alpha} \times \left(1 - \frac{t}{\lambda} \right)^{-1}$$

$$\text{Mean} = C_x'(t)$$

$$\begin{aligned}
C_x(t) &= \ln M_x(t) \\
&= \ln\left(e^{\lambda\alpha} \cdot \left(1-\frac{t}{\lambda}\right)^{-\alpha}\right) \\
&= \lambda\alpha + (-\alpha)\ln\left(1-\frac{t}{\lambda}\right) \\
&= \lambda\alpha - \alpha\ln\left(1-\frac{t}{\lambda}\right)
\end{aligned}$$

$$\begin{aligned}
C_x'(t) &= \frac{-\alpha}{\left(1-\frac{t}{\lambda}\right)} \cdot -\frac{1}{\lambda} \\
&= \frac{\alpha}{\lambda}\left(1-\frac{t}{\lambda}\right)^{-1}
\end{aligned}$$

$$C_x'(0) = \frac{\alpha}{\lambda}$$

$$\text{Mean} = \frac{\alpha}{\lambda}$$

$$\begin{aligned}
C_x''(t) &= -\frac{\alpha}{\lambda}\left(1-\frac{t}{\lambda}\right)^{2} \cdot -\frac{1}{\lambda} \\
&= \frac{\alpha}{\lambda^2}\left(1-\frac{t}{\lambda}\right)^{2}
\end{aligned}$$

$$C_x''(0) = \frac{\alpha}{\lambda^2}$$

Q4) $G_V(t) = \left(\frac{p}{1-qt}\right)^k$ $\quad p+q=1$

$G_V(t) = p^k(1-qt)^{-k}$ $\quad$ find : $P(V=4)$

$\cancel{= p^1(1-qt)^{-1} + p^2(1-qt)^{-2} + p^3(1+qt)^{-3} + \ldots}$

$= \Sigma\, p^k + \Sigma \binom{-k}{x} (1)^{-k} (qt)^{-k}$

$= p^k\, \Sigma \binom{v+k-1}{v-1} (qt)^k$

$=$

Q5) $f(x,y) = ax(x+y)$ $\quad 0<x<5$

$10<y<20$

i.e $30<3y<60$

ii) $P(Y > \frac{1}{3}X + 10) = P(Y - \frac{1}{3}X > 10)$

$= P\left(\frac{3Y - X}{3} > 10\right)$

$= P(3Y - X > 30)$

$= \int_0^5 \int_{30}^{60-5} ax(x+y)\,dy\,dx$

$= \int_0^5 \int_{30}^{55} ax(x+y)\,dy\,dx$

$= \int_0^5 \left[ax^2 + \frac{axy^2}{2}\right]_{30}^{55} dx$

$= \int_0^5 ax^2 + ax\left[\frac{55^2}{2} - \frac{30^2}{2}\right] dx$

$= \cancel{1062.5} \int_0^5 ax^2 + 1062.5\,ax\,dx$

$= \left[\frac{ax^3}{3} + \frac{1062.5\,ax^2}{2}\right]_0^5$

$= 41.667a + 13281.25a$

$= 13322.91a$

$= \cancel{6.}\ \cancel{0.6951083}\ \frac{13322.91}{2291.667}$

Q5) i) $f(x,y) = ax(x+y)$

$$\int_0^5 \int_{10}^{20} ax(x+y)\,dy\,dx = 1$$

$$\int_0^5 \int_{10}^{20} (ax^2 + axy)\,dy\,dx = 1$$

$$\int_0^5 \left[ax^2y + \frac{axy^2}{2}\right]_{10}^{20} = 1$$

$$\int_0^5 10ax^2 + ax(150)\,dx = 1$$

$$\left[\frac{10ax^3}{3} + 75ax^2\right]_0^5 = 1$$

$$a\left[\frac{1250}{3} + 1875\right] = 1$$

$$a = \cancel{\frac{1}{1916.6667}} \quad \frac{1}{2291.6667} = 4.36 \times 10^{-4}$$

$$E(Y/X) = \int_{10}^{20} \frac{f(x,y)}{f(x)} \times y\,dy$$

$$f(x) = \int_{10}^{20} ax(x+y)\,dy$$

$$= \left[ax^2y + \frac{axy^2}{2}\right]_{10}^{20}$$

$$= 10ax^2 + 150ax$$

$$\cancel{E(x/y)} = \int_{10}^{20} y \frac{f(x,y)}{f(x)} dy$$
$E(Y/X)$

$$= \int_{10}^{20} \left[y \times \frac{ax(x+y)}{10ax^2+150ax} dy \right]$$

$$= \int_{10}^{20} \left[y \frac{\cancel{ax}(x+y)}{\cancel{ax}(10x+150)} dy \right]$$

$$= \int_{10}^{20} \frac{xy+y^2}{(10x+150)} dy$$

$$= \frac{1}{10x+150} \left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_{10}^{20}$$

$$= \frac{1}{10x+150} \left[150x + 2333.33 \right]$$

$$= \frac{15x + 233.33}{x + 15}$$

$X \sim \text{Poisson}(\lambda)$, $Y \sim \text{Poi}(\mu)$

Q6) Arthur : $X - Y \sim \text{Poisson}(\lambda - \mu)$
Christine : $X + Y \sim \text{Poisson}(\lambda + \mu)$

Arthur : let $Z = X - Y$

$M_X(t) = e^{\mu(e^t - 1)}$
$M_Y(t) = e^{\lambda(e^t - 1)}$

$$M_Z(t) = \cancel{M_X}\ E(e^{tz})$$
$$= E(e^{t(x-y)})$$
$$= E(e^{tx}) \cdot E(e^{-ty})$$
$$= M_X(t) \cdot M_Y(-t)$$
$$= e^{\mu(e^t - 1)} \cdot e^{\lambda(e^{-t} - 1)}$$
$$= e^{(\mu + \lambda)(e^t - 1)}$$
$$= e^{(\lambda - \mu)(e^t - 1)}$$

By uniqueness property,
$\cancel{X+Y}$ $Z = X - Y \sim$ ~~Poi(μ − λ)~~ $\text{Poi}(\lambda - \mu)$ $\therefore$ Arthur is ~~wrong~~ correct

Christine : let $W = X + Y$
$M_X(t) = e^{\mu(e^t - 1)}$, $M_Y(t) = e^{\lambda(e^t - 1)}$

$$M_W(t) = E(e^{tW}) = E(e^{t(x+y)})$$
$$= E(e^{tx}) \cdot E(e^{ty})$$
$$= M_X(t) \cdot M_Y(t)$$
$$= e^{\mu(e^t - 1)} \cdot e^{\lambda(e^t - 1)}$$
$$= e^{(\lambda + \mu)(e^t - 1)}$$

By uniqueness property,
$(W = X + Y) \sim \text{Poi}(\mu + \lambda)$ $\therefore$ Christine is correct.

Q7.

i) $P(X/Y=2) = \sum \dfrac{P(X=x, Y=2)}{P(Y=2)}$

$= \dfrac{P(X=1, Y=2)}{P(\cancel{X}=2)} + \dfrac{P(X=2, Y=2)}{P(\cancel{X}=2)} + \dfrac{P(X=3, Y=2)}{P(\cancel{X}=2)} + \dfrac{P(X=4, Y=2)}{P(\cancel{X}=2)}$

$= 0 + \dfrac{0.3}{0.6} + \dfrac{0.1}{0.6} + \dfrac{0.2}{0.6}$

$= 0 + 0.5 + 0.16667 + 0.3333$

x	1	2	3	4
$P(X/Y=2)$	0	0.5	0.16667	0.333

$E(X/Y=2) = \sum x\, P(x/Y=2)$

$= 1\times 0 + 2\times 0.5 + 3\times 0.16667 + 4\times 0.333$

$= 2.83321$

~~V(X)~~ $E(X^2/Y=2) = \sum x^2 P(x/Y=2)$

$= 1\times 0 + 2^2\times 0.5 + 3^2\times 0.16667 + 4^2\times 0.333$

$= 8.82803$

$V(X/Y=2) = E(X^2/Y=2) - [E(X/Y=2)]^2$

$= 8.82803 - (2.83321)^2$

$= 0.80095$

ii) $f(u,v) = \frac{48}{67}(2uv - u^2) \qquad 0<u<1$

$\frac{u}{2} < v < 2$

$E(U/V=v) = ?$

$$f(v) = \frac{48}{64}\int_0^1 2uv - u^2 \, du$$

$$= \frac{48}{64} \times \left[\frac{\cancel{2}u^2v}{\cancel{2}} - \frac{u^3}{3}\right]_0^1$$

$$= \frac{48}{64} \times \left[v - \frac{v^3}{3}\right]$$

$$= \frac{48}{64} \cancel{u}(\cancel{u-1})(\cancel{1-u})\left(v - \frac{1}{3}\right)$$

$$E(U/V=v) = \int_0^1 u \times \frac{f(u,v)}{f(v)} \, du$$

$$= \int_0^1 u \times \frac{\frac{\cancel{48}\,u(2v-1)}{\cancel{67}}}{\frac{\cancel{48}\,u}{\cancel{67}}\cancel{(1-u)}\left(v - \frac{1}{3}\right)} \, du$$

$$= \left(v - \frac{1}{3}\right)^{-1} \int_0^1 2u^2v - u^3 \, du$$

$$= \left(v - \frac{1}{3}\right)^{-1} \left[\frac{2u^3v}{3} - \frac{u^4}{4}\right]_0^1$$

$$E(U/V=v) = \left(v - \frac{1}{3}\right)^{-1} \left[\frac{2v}{3} - \frac{1}{4}\right]$$

Q8. $\alpha = 3$, $\lambda = 2$

$X \sim Gamma(\alpha = 3, \lambda = 2)$

$E(X) = \frac{\alpha}{\lambda} = 1.5$ $\quad V(X) = \frac{\alpha}{\lambda^2} = 0.75$

$E(Y/X = x) = 3x + 1$

$V(Y/X = x) = 2x^2 + 5$

$E(Y) = E(E(Y/X = x))$

$= E(3x + 1)$

$= 3E(X) + 1$

$= 3 \times \frac{\alpha}{\lambda} + 1$

$= 3 \times \frac{3}{2} + 1$

$= \cancel{3.25}\ 5.5$

$V(Y) = E(V(Y/X = x)) + V(E(Y/X = x))$

$= E(2x^2 + 5) + V(3x + 1)$

$= 2E(X^2) + 5 + 9V(X)$

$= 2[V(X) + E(X)] + 5 + 9V(X)$

$= 2[0.75 + 1.5] + 5 + 9(0.75)$

$= 16.25$

Q9. $X \Rightarrow \mu_X = 3 \quad \sigma_x = 2$

$Y \Rightarrow \mu_Y = 4 \quad \sigma_Y = 1$

$\rho = -0.3 \qquad Z = X + Y$

$$\rho = \frac{Cov(X,Y)}{\sigma_x \sigma_y}$$

$$-0.3 = \frac{Cov(X,Y)}{2 \times 1}$$

$$Cov(X,Y) = -0.6$$

$E(XY) - E(X)E(Y) = -0.6$

$E(XY) - 3 \times 4 = -0.6$

$E(XY) = 12 - 0.6$

$= 11.4$

$$\begin{aligned}
Cov(X,Z) &= E(XZ) - E(X)E(Z) \\
&= E(X(X+Y)) - E(X)E(X+Y) \\
&= E(X^2 + XY) - E(X)E(X+Y) \\
&= V(X) + E(X)^2 + E(XY) - E(X)E(X).E(Y) \\
&= 2^2 + 3^2 + 11.4 - 3 \times 3 \times 4 \\
&= 4 + 9 + 11.4 - 36 \\
&= -11.6
\end{aligned}$$

$$\begin{aligned}
Var(\cancel{Z}) &= Var(X+Y) \\
&= Var(X) + Var(Y) + 2Cov(X,Y) \\
&= 2^2 + 1^2 + 2(-0.6) \\
&= 5 - 1.2 \\
&= 3.8
\end{aligned}$$

Q10. p.d.f $= k x^{-\alpha} e^{-y/\beta}$ $\alpha > 1,\ \beta > 0,\ k$ const.

$1 < x < \infty$

$1 < y < \infty$

$$\int_1^\infty \int_1^\infty k x^{-\alpha} e^{-y/\beta}\, dx\, dy = 1$$

$$\cancel{\int_1^\infty \int_1^\infty} \quad \int_1^\infty k e^{-y/\beta} \left[\frac{x^{-\alpha+1}}{-\alpha+1} \right]_1^\infty dy = 1$$

$$\int_1^\infty k e^{-y/\beta} \frac{x^{-1\times 10-\infty}}{-1\times 10^{-\infty}} dy = 1$$

$$\int_1^\infty k e^{-y/\beta} \times 0 \; dy = 1$$

$\therefore\ k = 0$