PROBABLITY AND STATISTICS
ASSIGNMENT-2

1] Let X be the amount of home insurance claim
Y be the amount of car insurance claim

$X \sim N(800, 100^2)$ $\quad Y \sim N(1900, 300^2)$

we require

$P((Y_1 + Y_2 + Y_3) > (X_1 + X_2 + X_3 + X_4) + 80)$

$= P((Y_1 + Y_2 + Y_3 - (X_1 + X_2 + X_3 + X_3)) > 800)$

We ~~me~~ need distribution of $(Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4)$:

$\sim N(3 \times 1200 - 4 \times 300, 3 \times 300^2 + 4 \times 100^2)$

$\sim N(400, 310000)$

$= P\left(Z > \frac{(300-400)}{\sqrt{310000}}\right)$

$P(Z > 0.8718)$

$1 - P(Z < 0.718) = 0.230$

3] i) $f(x) = \lambda e^{-\lambda(x-\alpha)}$,

Using the definition of MGF:

$M_X(t) = E[e^{tx}] = \int_{\alpha}^{\infty} e^{tx} \lambda e^{-\lambda(x-\alpha)} dx$

$= \int_{\alpha}^{\infty} \lambda e^{\lambda\alpha} \cdot e^{-(\lambda-t)x} dx$

$= \left[\frac{-\lambda e^{\lambda\alpha}}{\lambda - t} e^{-(\lambda-t)x} \right]_{\alpha}^{\infty}$

$= \frac{\lambda}{\lambda - t} e^{t\alpha}$, given $t < \lambda$

ii) $M_x(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} e^{tx}$

$M'_x(t) = \frac{1}{\lambda}\left(1 - \frac{t}{\lambda}\right)^{-1} e^{\alpha} + \alpha\left(1 - \frac{t}{\lambda}\right)^{-1} e^{t\alpha}$

$E(x) = M'_x(0) = \frac{1}{\lambda} + \alpha$

$M''_x(t) = \frac{2}{\lambda^2}\left(1 - \frac{t}{\lambda}\right)^{-3} e^{tx} + \frac{2\alpha}{\lambda}\left(1 - \frac{t}{\lambda}\right) + \alpha^2\left(1 - \frac{t}{\lambda}\right)^{-1} e^{tx}$

$E(x^2) = M''_x(0) = \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2$

$Var(x) = \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2 - \left(\frac{1}{\lambda} + \alpha\right)^2$

$Var(x) = \frac{1}{\lambda^2}$

4] Probablity, $P(X=n) = \frac{1}{n!} G_v^{(n)}(0)$

$P(V=4) = \frac{1}{4!} G_v^4(t)$

funding $G^4 v(t)$

$G' v(t) = k p^k q (1-qx)^{-k-1}$

$G'' v(t) = -k(-k-1) k p^k q^2 (1-qx)^{-k-2}$

$G''' v(t) = (-k-2)(-k-1) k p^k q^3 (1-qx)^{-k-3}$

$G'''' v(t) = -(k-3)(-k-2)(-k-1) k q^k q^4 (1-qx)^{-k-4}$

$G'''' v(0) = -(-k-3)(-k-2)(-k-1) k p^k q^4$

4. $f(x,y) = ax(x+y) \quad 0<x<5$

We know that,

$$\int_{10}^{20}\int_{0}^{5} ax(x+y)\,dx\,dy = 1$$

$$\int_{10}^{20}\int_{0}^{5} a(x^2+xy)\,dx\,dy = 1$$

$$\int_{10}^{20} a\left[\frac{x^3}{3} + \frac{x^2}{2}y\right]_0^5 dy = 1$$

$$\int_{10}^{20} a\left[\frac{125}{3} + \frac{25}{2}y\right] dy = 1$$

$$a\left[\frac{125}{3}y + \frac{25y^3}{4}\right]_{10}^{20} = 1$$

$$a\left[\frac{2500}{3} + 2500 - \frac{1250}{3} - 625\right] = 1$$

$$a = 0.00044$$

5] $E(Y|X) = \int_y y\frac{f(x,y)}{f(x)}\,dy$

$$f(x) = \int_y f(x,y)\,dx$$

$$= \int_{10}^{20} a(x^2+xy)\,dy$$

$$= a\left[x^2y + \frac{xy^2}{2}\right]_{10}^{20}$$

$$= a\left[20x^2 + 200x - 10x^2 - 50x\right]$$

$$= a\left[10x^2 + 150x\right]$$

$$E(Y/X) = \int_{10}^{20} y \frac{a[x^2+xy]}{a[10x^2+150x]} \, dy$$

$$\int_{10}^{20} y \frac{(x+y)}{(10x+150)} \, dy$$

$$= \frac{1}{10x+150} \left[\frac{y^2 x}{2} + \frac{y^3}{3} \right]_{10}^{20}$$

$$= \frac{1}{10x+150} \left(200x + \frac{3000}{3} - 50x - \frac{1000}{3} \right)$$

$$= \frac{1}{10x+150} \left(150x + \frac{7000}{3} \right)$$

d] Examining Arthur's statement,

$$X \sim Poi(\lambda) \qquad Y \sim Poi(\mu)$$

X has MGF : $M_X(t) = e^{\lambda(e^t-1)}$

Y has MGF : $M_Y(t) = e^{\mu(e^t-1)}$

X-Y has MGF : $M_X(t) \; M_Y(t)$
$$= e^{\lambda(e^t-1)} \, e^{-\mu(e^t-1)}$$
$$= e^{(\lambda-\mu)(e^t-1)}$$

It is the MGF of a Poison with parameter $(\lambda$

$$X-Y \sim Poi(\lambda-\mu)$$

So Arthur's statement is true

Similarly evaluating christine's statement

$$X \sim Poi(\lambda) \qquad Y \sim Poi(\mu)$$

$M_X(t) = e^{\lambda(e^t-1)}$ $\qquad$ $M_Y(t) = e^{\mu(e^t-1)}$

So $X+Y$ has MGF: $M_X(t)\, M_Y(t)$

$$= e^{\lambda(e^t-1)} e^{\mu(e^t-1)}$$

$$= e^{(\lambda+\mu)(e^t-1)}$$

It is a MGF of a poisson with parameters $(\lambda+\mu)$

$X+Y \sim Poi(\lambda+\mu)$

So Christine's statement is also true.

8] $X \sim Gamma(3, 2)$

$$E(Y|X=x) = 3x+1$$

$$Var(Y|X=x) = 2x^2+5$$

$$E(Y) = E(E(Y|X)) = E(3x+1)$$

$$= 3E(x)+1$$

But $E(X) = \frac{3}{2}$

$$E(Y) = 3(1.5)+1$$

$$= 5.5$$

Using $Var(Y) = E[Var(Y|X)] + Var[E(Y|X)]$

$$= Var(Y) = E(2x^2+5) + Var(3x+1)$$

$$= 2E(x^2) + 5 + 9\,Var(Y)$$

$$= 2\left((1.5)^2 + \frac{9}{16}\right) + 9\left(\frac{3}{4}\right)$$

$$Var(Y) = 12.375.$$

9] $E[X] = 3$

$\sigma_x = 2$

$E(Y) = 4$

$\sigma_Y = 1$

Correlation coefficient $= -0.3$

$Z = X+Y$

i) $Cov(X,Z) = Cov(X, X+Y)$

$= Cov(X,X) + Cov(X,Y)$

$= Var(X) + Cov(X,Y)$

$$Cov(X,Y) = -0.3 = \frac{Cov(X,Y)}{\sqrt{Var(X) \cdot Var(Y)}} = \frac{Cov(X,Y)}{\sqrt{4 \times 1}}$$

$Cov(X,Y) = -0.6$

$Cov(X,Z) = 4 - 0.6 = 3.4$

ii) $Var(Z) = Cov(Z,Z)$

$= Cov(X+Y, X+Y)$

$= Cov(X,X) + 2Cov(X,Y) + Cov(Y,Y)$

$= Var(X) + 2Cov(X,Y) + Var(Y)$

$= 4 + 2(3-0.6) + 1$

$= 3.8$

10] $k x^{-\alpha} e^{4/3}$ $\qquad 1 < x < \infty$

The PDF must intergrate to 1

$$\int_{y=1}^{\infty}\int_{x=1}^{\infty} k x^{-\alpha} e^{4/3} \, dx \, dy = 1$$

Integroding over x values given

$$\int_1^\infty K x^{-\alpha} e^{-y/\beta} = K e^{-y/\beta} \left[\frac{x^{-\alpha+1}}{-\alpha+1} \right]_1^\infty$$

$$= \frac{K e^{-y/\beta}}{\alpha - 1}$$

Integroding over y values

$$\int_1^\infty \frac{K e^{-y/\beta}}{\alpha-1} = \frac{K}{\alpha-1} \left[-\beta e^{-y/\beta} \right]_1^\infty$$

$$= \frac{K\beta e^{-1/\beta}}{\alpha - 1}$$

Equating to 1

$$\frac{K\beta e^{-1/\beta}}{\alpha - 1} = 1$$

$$K = \frac{(\alpha-1) e^{1/\beta}}{\beta}$$