

Statistic & Risk Modelling assignment.

Unit 3 & 4.

Qu 1.

$$y_t^2 - \beta_1 e^2 y_{t-1}^2 = 2(y_t - \beta_1 e^2 y_{t-1})\mu - (1 - \beta_1 e^2)\mu^2 + \beta_0 e^2$$

$$(i) \text{Cov}(y_t, y_{t-s})$$

$$y_t^2 - 2y_t\mu + \mu^2 = e_t^2 (\beta_0 + \beta_1 y_{t-1}^2 - 2\beta_1 y_{t-1}\mu + \beta_1 \mu^2)$$

$$(y_t - \mu)^2 = e_t^2 (\beta_0 + \beta_1 (y_{t-1} - \mu)^2)$$

$$y_t = \mu + e_t (\beta_0 + \beta_1 (y_{t-1} - \mu)^2)^{1/2}$$

$$E(y_t) = E(\mu) + E(e_t (\beta_0 + \beta_1 (y_{t-1} - \mu)^2)^{1/2})$$

$$E(y_t) = E(\mu) + 0$$

$$= \mu$$

$$\text{Cov}(y_t, y_{t-1}) = E(y_t y_{t-1}) - E(y_t)E(y_{t-1})$$

$$E_0 = E(\mu + e_t (\beta_0 + \beta_1 (y_{t-1} - \mu)^2)^{1/2})$$

$$(\mu + e_{t-s} (\beta_0 + \beta_1 (y_{t-s-1} - \mu)^2)^{1/2} - \mu^2$$

$$= E(\mu^2 + \mu e_t (\beta_0 + \beta_1 (y_{t-1} - \mu))^{\frac{1}{2}} + \mu e_{t-5} (\beta_0 + \beta_1 \dots \\ \dots (y_{t-5-1} - \mu)^2)^{\frac{1}{2}} + e_t (\beta_0 + \beta_1 (y_{t-1} - \mu))^{\frac{1}{2}} \times \\ e_{t-5} (\beta_0 + \beta_1 (y_{t-5-1} - \mu)^2)^{\frac{1}{2}} - \mu^2 \\ = \mu^2 - \mu^2 = 0$$

$$(ii) \text{Var}(y_t | y_{t-1}) = \text{var}(e_t) \text{var}(\beta_0 + \beta_1 (y_{t-1} - \mu)^2 \\ = \beta_0 + \beta_1 (y_{t-1} - \mu)^2$$

Var is dependent on y_{t-1}

y_t and y_{t-5} are independent

$$(iii) X_t = 0.5 y_t + 0.3t + 0.1$$

$$E(\Delta X_t) = E(X_t) - E(X_{t-1}) \\ = E(0.5 y_t + 0.3t + 0.1) - E(0.5 y_{t-1} + 0.3(t-1) + 0.1) \\ = 0.5\mu + 0.3t + 0.1 - 0.5\mu - 0.3(t-1) - 0.1 = 0.3$$

Mean is constant

$$2. (i) \quad y_t = y_{t-1} + 0.5y_{t-2} - 0.5y_{t-3} + 2 + 0.32_{t-1}$$

Characteristic eqn.

$$1 - z - 0.5z^2 + 0.5z^3 = 0$$

$$X = (1 - \beta)y$$

$$X_t - 0.5X_{t-2} = 2 + 0.32_{t-1}$$

$$\text{ARIMA}(2, 1)$$

$$y = \text{ARIMA}(2, 1, 1)$$

$$(ii) \quad X_t - 0.5X_{t-2} \\ (1 - 0.5z^2)$$

$$\text{Roots} = \pm\sqrt{2}$$

$$> 1$$

process is stationary

$$(iii) \quad X_t = 0.5X_{t-2} + 2 + 0.32_{t-1}$$

$$\begin{aligned} \text{Cov}(X_t, Z_t) &= \text{Cov}(0.5X_{t-2} + 2 + 0.32_{t-1}, Z_t) \\ &= 0 + \sigma^2 = \sigma^2 \end{aligned}$$

$$\begin{aligned} \text{cov}(X_t, Z_{t-1}) &= \text{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, Z_{t-1}) \\ &= 0 + 0 + 0.3\sigma^2 = 0.3\sigma^2 \end{aligned}$$

$$\begin{aligned} \text{cov}(X_t, Z_{t-2}) &= \text{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, Z_{t-2}) \\ &= 0 + 0 + 0.5\sigma^2 = 0.5\sigma^2 \end{aligned}$$

$$Y_0 = \text{cov}(X_t, X_t)$$

$$= \text{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, X_t)$$

$$= 0.5\sigma^2 + \sigma^2 + 0.09\sigma^2$$

$$= 0.5Y_2 + 1.09\sigma^2 \quad \text{--- (1)}$$

$$Y_1 = \text{cov}(X_t, X_{t-1})$$

$$= \text{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, X_{t-1})$$

$$= 0.5\sigma^2 + 0 + 0.3\sigma^2 \quad \text{--- (2)}$$

$$Y_k = \text{cov}(X_t, X_{t-k})$$

$$= \text{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, X_{t-k})$$

$$= 0.5Y_{k-1} \quad k \geq 2$$

$$Y_2 = \text{cov}(X_t, X_{t-2}) = \text{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, X_{t-2})$$

$$= 0.5Y_0 \quad \text{--- (3)}$$

Replace (3) into (1)

$$Y_0 = 0.25 Y_0 + 1.095^2$$

$$Y_0 = \frac{1.095^2}{0.75}$$

$$Y_1 = \frac{30^2}{5}$$

$$R_1 = \frac{Y_1}{Y_0} = \frac{45}{109}$$

From (3) and (4)

$$L(k) = 0.5 e^{(k-1)}$$

$$L(k) = 0.5$$

$$3. (1) a) Y_t = 0.4 Y_{t-1} + 0.2 Y_{t-2} + Z_t + 0.025 Z_{t-1} + 0.016$$

$$(1 - 0.4B - 0.2B^2) Y_t = (1 + 0.025B) Z_t + 0.016$$

Characteristic : $1 - 0.4\lambda - 0.2\lambda^2 = 0$

$$d=0$$

ARIMA (2, 0, 1) process

$$(b) (1 - 0.4 - 0.2) / (F(Y)) = 0.016$$

$$E(Y_t) = \frac{0.016}{0.4} = 4\%$$

(c) Roots: $-1 \pm \sqrt{6}$
 1.4495 , -3.4495
 > 1

y_t is a stationary process

ii) (a)

(b) Inspection of graph of time plot of residual.
 (cluster or ~~new~~ fluctuations (unfit)).

Inspection of sample autocorrelation function of residuals.

4. $y_t = -\alpha y_{t-1} + \alpha^2 y_{t-2} + z_t$

(i) AR(2) process

$$(1 + \alpha B + \alpha^2 B^2) y_t = z_t$$

$$1 + \alpha \lambda - \alpha^2 \lambda^2 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2\alpha}$$

$$\lambda > 1$$

$$\alpha < \frac{\sqrt{5} - 1}{2}$$

$$(ii) y_t = -\alpha y_{t-1} + \alpha^2 y_{t-2} + \varepsilon_t$$

$$\sigma_0 \text{ cov}(y_t, y_t) = -\alpha y_1 + \alpha^2 y_2 + \sigma^2 \quad \text{--- (1)}$$

$$\sigma_1 = \text{cov}(y_t, y_{t-1}) = -\alpha y_0 + \alpha^2 y_1 \quad \text{--- (2)}$$

$$\sigma_2 = \text{cov}(y_t, y_{t-2}) = -\alpha \sigma_1 + \alpha^2 y_0 \quad \text{--- (3)}$$

from (2)

$$\sigma_1 = \frac{-\alpha y_0}{1 - \alpha^2} \quad \text{--- (4)}$$

Replace (4) into (3)

$$\sigma_2 = -\alpha \left(\frac{-\alpha y_0}{1 - \alpha^2} \right) + \alpha^2 y_0$$

$$\frac{(2\alpha^2 - \alpha^4) \cdot y_0}{1 - \alpha^2} \quad \text{--- (5)}$$

$$\sigma_0 = \frac{\sigma^2 (1 - \alpha^2)}{1 - 2\alpha^2 - 2\alpha^4 + \alpha^6}$$

$$\sigma_1 = \frac{\sigma^2 (-\alpha)}{1 - 2\alpha^2 - 2\alpha^4 + \alpha^6}$$

$$\sigma_2 = \frac{\sigma^2 (2\alpha^2 - \alpha^4)}{1 - 2\alpha^2 - 2\alpha^4 + \alpha^6}$$

$$5. (i) (X_n - X_{n-1}) = \alpha (X_{n-1} - X_{n-2}) + \epsilon_n$$

$$\alpha = \rho_1 = \frac{\sum_{i=2}^{200} (X_i - X_{i-1})(X_{i-1} - X_{i-2})}{\sum_{i=2}^{200} (X_i - X_{i-1})^2}$$

$$\alpha = \frac{587.83}{936.47} = 0.6277$$

$$\gamma_0 = \frac{1}{200} \sum_{i=2}^{200} (X_i - X_{i-1})^2$$

$$= \frac{936.49}{200} = 4.6825$$

$$\gamma_0 = \frac{\sigma^2}{1 - \rho^2} = \frac{\sigma^2}{1 - \alpha^2}$$

$$\hat{\sigma}^2 = 1 - \hat{\alpha}^2 \gamma_0$$

$$(1 - 0.6277^2) \cdot 4.6825 = 2.8376$$

(ii) X_{201}

$$(\hat{X}_{200}(1) - X_{200}) = \hat{\alpha} (X_{200} - X_{199})$$

$$= 0.6277 (1.93 - 0.82) = 0.6967$$

$$\hat{X}_{200}(1) = X_{200} + 0.6967$$

$$= 1.93 + 0.6907 = 2.6267$$

6. (i) Forecasting future values of process
 Description of the data
 Construction of a model which fits the data
 Deciding whether the process is out of control
 and Required action.

(ii) $AR(1) \rightarrow \text{Yes}$

$MA(1) \rightarrow \text{No}$

$AR(2) \rightarrow \text{No}$

Random walk $\rightarrow \text{Yes}$

(iii) $X_t = a + bt + Y(t)$

$$E(X_t) = a + bt + E(Y(t))$$

Y_t is stationary

$$E(Y_t) = \text{constant}$$

a. $E(X_t) = a + bt + c$ which depends on t

So not stationary

b. $E(\nabla X_t) = E(X_t) - E(X_{t-1})$

$$a + bt + c - (a + b(t-1) + c) = b$$

Mean does not depend on t

$$(iv) X_1(t) - X_2(t)$$

$$= aX_1(t-1) - X_2(t-1) + b(X_2(t-1) - X_1(t-1)) + e_1(t) - e_2(t)$$

$$= (a-b) [X_1(t-1) - X_2(t-1)] + (e_1(t) - e_2(t))$$

$$Y_t = X_1(t) - X_2(t)$$

$$Y_t = AR(1) \text{ process}$$

7. i) Autoregressive process (AR)

~~The~~ is An autoregressive process with order p is a sequence of random variable (X_t) defined consecutively by the rule:

$$X_t = \mu + \alpha_1(X_{t-1} - \mu) + \alpha_2(X_{t-2} - \mu) + \dots + \alpha_p(X_{t-p} - \mu) + e_t$$

Thus, the AR(p) model try to explain the current value of X as a linear combination of past values with some additional externality generated random variable.

(b) Moving average process (MA)

A moving average process with order q is a sequence defined by

$$X_t = \mu + e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \dots + \beta_q e_{t-q}$$

The MA(q) model explains the relationship b/w X_t as an indirect effect arising from the fact that the current value of the process results from recently past random errors.

(c) Autoregressive moving average process (ARMA)

The two basic processes (AR and MA) are combined together to give (ARMA)

$$X_t = \mu + \alpha_1 (X_{t-1} - \mu) + \alpha_2 (X_{t-2} - \mu) + \dots + \alpha_p (X_{t-p} - \mu) + e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \dots + \beta_q e_{t-q}$$

(ii) a) MA(1) $\rightarrow$ stationary ARIMA(0,0,1)

(b) ARMA(2,3)

Cannot be differenced

$\Rightarrow$ ARIMA(2,0,3)

$$(1 - 1.4B^2)X_t = e_t + 0.5e_t$$

$$\lambda = \pm 0.8452$$

$$\lambda \leq 1$$

(c) ARMA(2, 1)

$$X_t - 1.4X_{t-1} + 0.4X_{t-2} = e_t - e_{t-1}$$

$$(X_t - X_{t-1}) = 0.4(X_{t-1} - X_{t-2}) = e_t - e_{t-1}$$

$$\nabla X_t = 0.4 \nabla X_{t-1} = e_t - e_{t-1}$$

ARIMA(, , ,)

(ii) $\gamma_0 = \text{cov}(X_t, X_t)$

$$= \text{cov}(e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3}, e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3})$$

$$= \sigma^2 + 0.25^2 \sigma^2 + 0.5^2 \sigma^2 + 0.25^2 \sigma^2$$

$$= 1.375 \sigma^2$$

$\gamma_1 = \text{cov}(X_t, X_{t-1})$

$$= \text{cov}(e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3}, e_{t-1} + 0.25e_{t-2} + 0.5e_{t-3} + 0.25e_{t-4})$$

$$= 0.25 \sigma^2 + 0.5(0.25) \sigma^2 + 0.5(0.25) \sigma^2$$

$$= 0.5 \sigma^2$$

$$\gamma_2 = \text{cov}(X_t, X_{t-2})$$

$$\text{cov}\left(\epsilon_t + 0.25\epsilon_{t-1} + 0.5\epsilon_{t-2} + 0.25\epsilon_{t-3}, \epsilon_{t-2} + 0.25\epsilon_{t-3} + \right.$$

$$\left. 0.5\epsilon_{t-4} + 0.25\epsilon_{t-5}\right)$$

$$= 0.5\sigma^2 + 0.25^2\sigma^2 = 0.5625\sigma^2$$

$$\gamma_3 = \text{cov}(X_t, X_{t-3})$$

$$\text{cov}\left(\epsilon_t + 0.25\epsilon_{t-1} + 0.5\epsilon_{t-2} + 0.25\epsilon_{t-3}, \epsilon_{t-3} + 0.25\epsilon_{t-4} + \right.$$

$$\left. 0.5\epsilon_{t-5} + 0.25\epsilon_{t-6}\right)$$

$$= 0.25\sigma^2$$

$$\gamma_k = 0 \quad k > 3$$

$$\rho_0 = 1$$

$$\rho_1 = 0.364$$

$$\rho_2 = 0.404$$

$$\rho_3 = 0.183$$

$$\rho_k = 0 \quad k > 3$$

$$2. (i) X_t = A_t + B_t$$

$$A_t = 0.5 A_{t-1} + 0.5 B_{t-1} + e_t^{(A)}$$

$$B_t = 0.7 A_{t-1} - 0.7 A_{t-2} + e_t^{(B)}$$

$$Y_t = \begin{pmatrix} A_t \\ B_t \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.7 & 0 \end{pmatrix} \begin{pmatrix} A_{t-1} \\ B_{t-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -0.7 & 0 \end{pmatrix} \begin{pmatrix} A_{t-2} \\ B_{t-2} \end{pmatrix} + \begin{pmatrix} e_t^{(A)} \\ e_t^{(B)} \end{pmatrix}$$

$$Y_t = \begin{pmatrix} A_t \\ B_t \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.7 & 0 \end{pmatrix} Y_{t-1} + \begin{pmatrix} 0 & 0 \\ -0.7 & 0 \end{pmatrix} Y_{t-2} + \begin{pmatrix} e_t^{(A)} \\ e_t^{(B)} \end{pmatrix}$$

$$\lambda^2 - 0.8\lambda - 0.35 = 0$$

$$\lambda < 1$$

$$Y_t \Rightarrow \text{stationary}$$

(ii) (a)

$$X_t = (\alpha + 1) X_{t-1} - (\alpha + 0.25\alpha^2) X_{t-2} + 0.25\alpha^2 X_{t-3} + e_t$$

$$X_t - X_{t-1} = \alpha (X_{t-1} - X_{t-2}) - 0.25\alpha^2 (X_{t-2} - X_{t-3}) + e_t$$

$$Y_t = \alpha Y_{t-1} - 0.25\alpha^2 Y_{t-2} + e_t$$

$$X_t = \text{ARIMA}(2, 1, 0)$$

$$(1 - \alpha\beta + 0.25\alpha^2\beta^2)Y_t = e_t$$

$$\beta = \frac{2}{\alpha}$$

$$\lambda > 1$$

$$\alpha < 2$$

ARIMA (2,1,2)

$$(b) \text{Cov}(Y_t, e_t) = \text{Cov}(e_t, e_t) = \sigma^2$$

$$\gamma_0 = \text{Cov}(Y_t, Y_t)$$

$$\alpha\gamma_1 - 0.25\alpha^2\gamma_2 + \sigma^2 = (1) \quad \text{--- (1)}$$

$$\gamma_1 = \alpha\gamma_0 - 0.25\alpha^2\gamma_1 \quad \text{--- (2)}$$

$$\gamma_2 = \alpha\gamma_1 - 0.25\alpha^2\gamma_0 \quad \text{--- (3)}$$

$$\gamma_k = \alpha\gamma_{k-1} - 0.25\alpha^2\gamma_{k-2}$$

from (2) $\gamma_1 = \frac{\alpha\gamma_0}{1 + 0.25\alpha^2} \quad \text{--- (4)}$

Replace (4) into (1)

$$\gamma_0 = \alpha\gamma_1 - 0.25\alpha^2(\alpha\gamma_1 - 0.25\alpha^2\gamma_0) + \sigma^2$$

$$1 - (0.25\alpha^2)^2 \gamma_0 = \frac{\alpha^2(1 - 0.25\alpha^2)}{1 + 0.25\alpha^2} \gamma_0 + \sigma^2$$

$$\gamma_0 = \frac{1 + 0.25\alpha^2}{(1 - 0.25\alpha^2)^3} \alpha^2$$

10. (i) $p=1$ $q=4$
ARMA (1, 4)

(ii) Non-linear non stationary time series model includes:

Bilinear models are those that exhibit bursty behavior

$$X_n = \alpha (X_{n-1} - \mu) - \mu + \epsilon_n + \beta \epsilon_{n-1} + \phi (X_{n-1} - \mu) \epsilon_{n-1}$$

Threshold autoregressive model are used to model cyclical pattern behavior

(iii)

$$X_n = \mu + \alpha_1 (X_{n-1} - \mu) + \epsilon_n$$

$$X_n = \mu + \alpha_2 (X_{n-1} - \mu) + \epsilon_n$$

(iii) $X_t = e_t + \beta e_{t-1}$

$$\text{Var}(X_t) = \text{Var}(e_t + \beta e_{t-1})$$

$$\text{Var}(e_t) + \beta^2 \text{Var}(e_{t-1})$$

$$\sigma^2 + \beta^2 \sigma^2$$

$$(1 + \beta^2) \sigma^2$$

$$\Delta Y_t = (0.6 + 0.3t + X_t) - (0.6 + 0.3(t-1) + X_{t-1})$$

$$= 0.3 + (X_t - X_{t-1})$$

11. (i) Description of data

forecasting future data values

deciding whether the process is out of control

Construction of a model which fits data.

(ii) Univariate

It is a sequence of obs (X_t) recorded at regular interval. The state space is continuous but time is discrete.

Invertibility

A process is invertible if we can write white noise term as a convergent sum of X terms

Markov

A time series has markov property if:

$$P(X_t = a | X_1 = x_1, \dots, X_{t-1} = x_{t-1}) = P(X_t = a | X_{t-1} = x_{t-1})$$

- (iii)
1. Seasoning differencing.
 2. Method of moving average.

1. Least square trend model
2. Differencing

12. (i) By presence of deterministic effect
(deterministic trend is seasonal effect)
It process is the integrated version of a
more fundamental process.

(ii) a) ARIMA (p, d, z)

$$(1-B)^d \phi(B)(X_t - \mu) = \theta(B)e_t$$

(b) Tentative identification of a model for ARIMA
Estimation of the parameter
Diagnostic checker

(c) $d=1$

(d) (i) $X_t = 0.8e_{t-1} + e_t$

MA(1)

ARIMA(0,0,1)

(ii) $X_t = 2X_{t-1} + e_t + 0.5e_{t-4}$

(iii) $X_t = 1.5X_{t-1} + 0.2X_{t-2} + e_{t-1} + e_{t-2}$

ARMA(2,1)

$$X_t - 1.5X_{t-1} - 0.2X_{t-2} = e_t + e_{t-2}$$

13(i) A weakly stationary process has constant mean and covariance is constant for each lag

(ii) $X_n = Z_n + \beta Z_{n-1}$

$$E(X_n) = (1+\beta)\mu$$

$$\begin{aligned} \text{Var}(X_n) &= \text{Var}(Z_n + \beta Z_{n-1}) \\ &= \text{Var}(1+\beta^2)\sigma^2 \end{aligned}$$

$$(iii) X_n = 1.5X_{n-1} - 0.5X_{n-2} + 2$$

$$(1 - 1.5B + 0.5B^2)X = 2$$

$$(1-B)(1-0.5B)$$

$$1 = b$$

$$B = \frac{1}{0.5}$$

not stationary

$$(iv) X = ARIMA(1, 1, 0)$$