Probability and Statistics
Chapter: Unit 1 and 2
Assignment 1

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Q1 i) 4, 7, 9, 9, 11, 15, 18, 18, 18, 25

$$\text{Mean} = \frac{\text{Sum of all observations } (\Sigma x)}{\text{No. of all observations } (n)}$$

$$= \frac{\Sigma x}{n}$$

$$= \frac{134}{10}$$

$$= 13.4$$

$$\text{Median} = \left(\frac{n+1}{2}\right)^{th} \text{observation}$$

$$= \left(\frac{11}{2}\right)^{th} \text{ to observation}$$

$$= \left(\frac{11+15}{2}\right)$$

$$= 13$$

Mode = observation with maximum frequency
= 18

standard deviation = $\sqrt{\frac{\Sigma(x_i-\mu)^2}{N}}$

x_i	$x_i-\mu$	$(x_i-\mu)^2$
4	-9.4	88.36
7	-6.4	40.96
9	-4.4	19.36
9	-4.4	19.36
11	-2.4	5.76
15	1.6	2.56
18	4.6	21.16
18	4.6	21.16
18	4.6	21.16
25	11.6	134.56

$\Sigma : 355.04$
$+ 19.36$

$\Sigma(x_i-\mu)^2 = 374.4$

$\frac{\Sigma(x_i-\mu)^2}{N} = 37.44$

standard deviation $= \sqrt{\frac{\Sigma(x_i-\mu)^2}{N}} = \sqrt{37.44}$

$= 6.118823$

$Q_1 = \left(\frac{n+1}{4}\right)^{th}$ observation $= 4$ $\frac{7+9}{2} = 8$

$Q_3 = \left(\frac{3n+1}{4}\right)^{th}$ observation $= \frac{18+18}{2} = 18$

Inter Quartile Range $= Q_3 - Q_1$

$= 18 - 8$

$= 10$

Q1 ii)

No. of households x	No. of people f	fx	c.f	$x-\bar{x}$	$(x-\bar{x})^2$	$f(x-\bar{x})^2$
0	5	0	5	-2	4	20
1	11	11	16	-1	1	11
2	26	52	42	0	0	0
3	15	45	57	1	1	15
4	3	12	60	2	4	12
	60	120			10	58

Mean $= \dfrac{\Sigma fx}{\Sigma f} = \dfrac{120}{60} = 2$

Median $= \left(\dfrac{n+1}{2}\right)^{th}$ observation $=$ ~~30st~~ 30.5^{th} term
$= 2$

Mode = highest frequency term
$= 2$

Standard deviation $= \sqrt{\dfrac{\Sigma f(x-\bar{x})^2}{N}} = \sqrt{\dfrac{58}{60}}$

$= 0.983192$

$Q_1 = \left(\dfrac{n+1}{4}\right)^{th}$ term $= 16^{th}$ term $= 1$

$Q_3 = \left(\dfrac{3n+1}{4}\right)^{th}$ term $= 46^{th}$ term $= 3$

Inter quartile range $= Q_3 - Q_1$
$= 3 - 1$
$= 2$

Q2) $\lambda = 0.5$ $\quad X \sim \text{Poisson}(\lambda = 0.5)$

i) $P(X=0) = \dfrac{e^{-\lambda} \times \lambda^{x}}{x!} = \dfrac{e^{-0.5} \times 0.5^{0}}{1}$

$= 0.606531$

ii) case (i) X_1 be claims for 3 consecutive years where one or more claims exist

P(1 or more claim in 1 yr & no claim in other 2 yrs)
= P(1 or more claim in 1st yr) × P(no claim in 2nd yr & 3rd yr)

+

P(1 or more claim in 2nd yr) × P(no claim in 1st & 3rd yr)

+

P(1 or more claim in 3rd yr) × P(no claim in 2nd & 1st yr)

$P(X \geq 1) = 1 - P(X < 1)$
$= 1 - P(X = 0)$
$= 1 - 0.606531 = 0.393469$

∴ required Probability =
$0.393469 \times 0.606531 \times 0.606531$
+
$0.393469 \times 0.606531 \times 0.606531$
+
$0.393469 \times 0.606531 \times 0.606531$

$= 0.434247843$

iii) P(2nd claim occurs at ~~2 or~~ more than 2 years)
= P(claim occured) × P(no claim in next yr 1) ×
P(no claim in next yr 2)

$= \frac{e^{-\lambda} \times \lambda^x}{x!} \times P(X=0) \times P(X=0)$

$= e^{-0.05} \times 0.05 \times 0.606531^2$

$= 0.017496907$

Q3 $X \sim Gamma(\alpha = 2, \lambda = 1/2)$

i) $E(X) = \frac{\alpha}{\lambda} = \frac{2}{1/2} = 4$

$V(X) = \frac{\alpha}{\lambda^2} = \frac{2}{(1/2)^2} = 8$

$\therefore$ standard distribution $= \sqrt{8} = 2.82843$

ii) Gamma distribution is positively skewed. This means that it has mode > median > mean.

iii) cumulative distribution function $F_X(X)$

$$F_X(X) = \int_0^x \left(\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \right) dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^x \left(x^{\alpha-1} e^{-\lambda x} \right) dx$$

$$= \frac{(1/2)^2}{\Gamma(2)} \int_0^{x} x e^{-1/2 x} \, dx$$

$$= \frac{1}{4} \int_0^{x} \left(x e^{-1/2 x} \right) dx$$

Using u·v rule for integration,

$$= \frac{1}{4} x \times \cancel{\Gamma(2)} \int_0^x e^{-1/2 x} \, dx - \int_0^x \frac{x}{4} \frac{d}{dx} \int_0^x e^{-1/2 x} \, dx \, dx$$

$$= \left[\frac{1}{4} x \times \frac{e^{-1/2 x}}{-1/2} \right]_0^x - \frac{1}{4} \int_0^x \frac{e^{-1/2 x}}{-1/2} \, dx$$

$$= \left[\frac{-1}{2} x \times \left(e^{-1/2 x} - 0 \right) \right] + \frac{1}{2} \left[\frac{e^{-1/2 x}}{-1/2} \right]_0^x$$

$$= \frac{-1}{2} x e^{-1/2 x} + 0 - e^{-1/2 x} + 1$$

$$= 1 - e^{-1/2 x} \left(\frac{x}{2} + 1 \right)$$

Q4

18 staff members

$\swarrow \qquad \searrow$

10M   8F

6M FQ   4M UQ   7F FQ   1F UQ

P(2 Qualified females/atleast one female)

$$= \frac{{}^{7}C_2}{({}^{8}C_2)\cancel{8} + ({}^{10}C_1 \times {}^{8}C_1 \times 2)}$$

$$= \frac{21}{188}$$

$$= 0.111702128$$

Q5.

$P(L1) = 0.4$ $P(L1/L) = 0.2$

$P(L2) = 0.5$ $P(L2/L) = 0.4$

$P(L3) = 0.1$ $P(L3/L) = 0.1$

$$P(L/L2) = \frac{P(L2/L) \times P(L2)}{P(L1/L) \times P(L1) + P(L2/L) \times P(L2) + P(L3/L) \times P(L3)}$$

$$= \frac{0.4 \times 0.5}{(0.2 \times 0.4) + (0.4 \times 0.5) + (0.1 \times 0.1)}$$

$$= \frac{0.20}{0.29}$$

$$= 0.68696552$$

Q6 $X \sim U(1,2)$

$Y = \dfrac{3}{6-X} \Rightarrow X = \dfrac{6Y-3}{Y}$

$F(X) = \dfrac{x-a}{b-a} \doteq \dfrac{x-1}{2-1} = x-1$

$F\left(\dfrac{6Y-3}{Y}\right) = \left(\dfrac{6Y-3}{Y}\right) - 1$

$= \dfrac{6Y-3-Y}{Y}$

$= \dfrac{5Y-3}{Y}$

$f\left(\dfrac{6Y-3}{Y}\right) = F'\left(\dfrac{6Y-3}{Y}\right)$

$= \dfrac{d}{dx}\left(5Y-3\right) \cdot Y^{-1}$

$= \left((5Y-3) \cdot -1 \cdot Y^{-2}\right) + Y^{-1}(5)$

$f\left(\dfrac{6Y-3}{Y}\right) = -\dfrac{5Y-3}{Y^2} + \dfrac{5}{Y}$

$f(x) = \dfrac{-5Y-3+5Y}{Y^2} = \dfrac{3}{Y^2}$

p.d.f of $x = \dfrac{3}{Y^2}$

$F_Y(y) = P(Y \leq y) = P\left(\dfrac{3}{6-x} \leq y\right)$

$= P\left(X \leq \dfrac{6y-3}{y}\right) = F\left(\dfrac{6y-3}{y}\right)$

$f_y(Y) = \int_0^{\frac{6y-3}{y}} \dfrac{3}{y^2}\, dy = \left[-\dfrac{3}{y}\right]_0^{\frac{6y-3}{y}} = -3\left[\dfrac{y}{6y-3}\right]$

$= \dfrac{-3y}{6y-3}$

Ans: $\dfrac{-y}{2y-1}$ $\qquad = \dfrac{-y}{2y-1}$

Q7

X	1	2	3	4	5
$P(X=x)$	0.05	0.15	0.35	0.4	0.05

i) $F(3) = P(X \leq 3)$

$= P(X=1) + P(X=2) + P(X=3)$

$= 0.05 + 0.15 + 0.35$

$= 0.55$

ii) $E(X) = \sum_{i=1}^{5} x\, P(X=x)$

$= 1\times0.05 + 2\times0.15 + 3\times0.35 + 4\times0.4 + 5\times0.05$

$= 0.05 + 0.3 + 1.05 + 1.6 + 0.25$

$= 3.25$

iii) $V(X) = ?$

$E(X^2) = \sum_{i=1}^{5} x^2\, P(X=x)$

$= 1^2\times0.05 + 2^2\times0.15 + 3^2\times0.35 + 4^2\times0.40 + 5^2\times0.05$

$= 11.45$

$V(X) = E(X^2) - [E(X)]^2$

$= 11.45 - (3.25)^2$

$= 0.8875$

iv) skewness = ?

$E(X^3) = \sum_{i=1}^{5} x^3\, P(X=x)$

$= 1^3\times0.05 + 2^3\times0.15 \ldots + 5^3\times0.05$

$= 42.55$

$$\text{skewness} = E(X-\bar{X})^3$$
$$= E(X^3) - 3E(X)E(X^2) + 2[E(X)]^3$$
$$= 42.55 - 3[3.25 \times 11.45] + 2[3.25]^3$$
$$= -0.43125$$

[negative skewness shows that the probability distribution is negatively skewed or Mean < Median < Mode.]

Q8 $P(X > 1000) = 0.2$ where X is random variable denoting insurance claims.

Sample = 15

P(fewer than 3 exceed 1000)

= P(1 exceed 1000) + P(2 exceed 1000) + P($\cancel{3}$ 0 exceed 1000)

$$= [0.2 \times 0.8^{14}] + [0.2^2 \times 0.8^{13}] + [0.2^{\cancel{3}\,0} \times 0.8^{15}]$$

$= \cancel{0.011544872}$ 0.046179488

Q9 Let X = winning a prize in a week

P(winning) = 0.1

$$P(3^{rd}\text{ prize}/\text{winning}) = \frac{P(3^{rd}\text{ prize} \cap \text{Winning})}{P(\text{winning})}$$
$$= \left(\frac{1}{3} \times 0.1\right) / 0.1 = \frac{1}{3}$$

$$P(\text{wins } 3^{rd}\text{ prize in } 7^{th}\text{ week}) = P(\text{losing in 6 weeks}) \times P(3^{rd}\text{ prize}/\text{winning})$$
$$= (1-0.1)^6 \times \frac{1}{3}$$
$$= (0.9)^6 \times 0.333$$
$$= 0.17697$$

Q10 $\lambda = 2/5 = 0.4$.

Let X be number of claims per day.

$X \sim \text{Poisson}(\lambda = 0.4)$

$P(X \geq 2) = P(\text{more than 2 claims per day})$

$= 1 - P(X \leq 2)$

Using Pg 175 of Actuarial Tables book

$= 1 - 0.99207$

$= 0.00793$

Q11 $P(\text{male child}) = P(\text{female child}) = 0.5$

5 boys, 2 girls

$P(\text{next 3 children are boys}) =$
$P(10^{th} \text{ child is } 8^{th} \text{ boy})$

X be number of children to a women

$X \sim \text{Negative Bin}(k = 8, p = 0.5)$, $x = 10$

$P(10^{th} \text{ child is } 8^{th} \text{ boy}) = {}^{x-1}C_{k-1}\, p^k q^{x-k}$

$= {}^{10-1}C_{8-1}\, p^8 q^{10-8}$

$= {}^{9}C_{7} (0.5)^8 (0.5)^2$

$= 36\ (0.5)^{10}$

$= 0.03515625$

Q12. $\lambda = 10$ per day

$P(\text{time between 2 claims}) = ?$

$T \sim Exp(\lambda = 10)$

$P(T < 0.1) = \int_0^{0.1} \lambda e^{-\lambda x} = F(0.1)$

$= [1 - e^{-\lambda t}]$

$= 1 - e^{-10 \times 0.1}$

$= 1 - 0.367879$

$= 0.6321206$

Q13. $X \sim U(a = 75, b = 120)$

$P(80 < x < 100) = ?$

$f(x) = \frac{1}{b-a} = \frac{1}{120-75} = \frac{1}{45}$

$P(80 < x < 100) = \int_{80}^{100} \frac{1}{45} dx$

$= \frac{1}{45} [x]_{80}^{100}$

$= \frac{1}{45} [100 - 80]$

$= \frac{20}{45}$

$= 0.4444$

Q14 $X \sim Gamma(5, 0.4)$

$\alpha = 5, \quad \lambda = 0.4$

$\therefore 2\lambda X \sim \chi^2_{2\alpha}$

$P(X < 22.89) = P(2\lambda X < 2 \times 0.4 \times 22.89)$

$= P(\chi^2_{2\alpha} < 18.312)$

$= P(\chi^2_{10} < 18.312)$

By linear Interpolation,

$x_1 = 18.0 \qquad f(x_1) = 0.9450$

$x_2 = 18.5 \qquad f(x_2) = 0.9529$

$x = 18.312 \qquad f(x) = ?$

$$\frac{x_2 - x_1}{x_1 - x} = \frac{f(x_2) - f(x_1)}{f(x_1) - f(x)}$$

$$\frac{18.5 - 18}{-18 + 18.312} = \frac{0.9529 - 0.9450}{f(x) - f(x_1)}$$

$$\therefore f(x) = \frac{0.312 \times 0.0079}{0.5} + 0.9450$$

$$= 0.9499296$$

$\therefore P(\chi^2_{10} < 18.312) = \cancel{1 - P(\chi^2_{10} \geq 18.312)}$

$= 0.9499296$

Q15) p.d.f : $f(x) = \dfrac{k}{(x+5)^4} \quad , x > 0$

i) By property of p.d.f,

$$\int_0^{\infty} \frac{k}{(x+5)^4}\, dx = 1$$

$$\int_0^{\infty} k\,(x+5)^{-4}\, dx = 1$$

$$k\left[\frac{(x+5)^{-3}}{-3}\right]_0^{\infty} = 1$$

$$k\left[\frac{6^{-3}}{-3} - \frac{5^{-3}}{-3}\right] = 1$$

$$\frac{k}{3}\left[\frac{-1}{216} + \frac{1}{125}\right] = 1$$

$$\frac{k}{3} \times \frac{91}{27000} = 1$$

$$k = \frac{81000}{91} = 890.1099$$

$$k\left[0 - \frac{5^{-3}}{-3}\right] = 1$$

$$k\left[\frac{1}{375}\right] = 1$$

$$k = 375$$

ii) $F(10) = \displaystyle\int_0^{10} \frac{k}{(x+5)^4}\, dx$

$$= 375 \int_0^{10} (x+5)^{-4}\, dx$$

$$= 375 \left[\frac{(x+5)^{-3}}{-3}\right]_0^{10}$$

$$= 375 \left[\frac{15^{-3}}{-3} - \frac{5^{-3}}{-3}\right]$$

ii) [Contd] $= 375 \left[-\frac{1}{10125} + \frac{1}{375} \right]$

$= \cancel{375} \left[\frac{-375 + 10125}{\cancel{375} \times 10125} \right]$

$= \frac{9750}{10125}$

$= 0.962963$

iii) $E(X) = \int x f(x) \, dx$

$= \int_{x=0}^{\infty} x \times \frac{375}{(x+5)^4} dx$

$= 375 \int_{0}^{\infty} \frac{x}{(x+5)^4} dx$

let $x+5 = t \Rightarrow x = t-5$
differentiate w.r.t x
$1 + 0 = \frac{dt}{dx}$

$dt = dx$

Upper limit:
$t = \infty + 5 = \infty$
Lower limit
$t = 0 + 5 = 5$

$E(X) = 375 \int_{5}^{\infty} \frac{t-5}{t^4} dt$

$= 375 \int_{5}^{\infty} (t-5) t^{-4} dt$

$= 375 \int_{5}^{\infty} (t^{-3} - 5t^{-4}) dt$

$$= 375 \cancel{\int_5^{\infty}} \left[\frac{t^{-2}}{-2} + \frac{5t^{-3}}{-3} \right]_5^{\infty}$$

$$= -375 \left[\frac{1}{t^2} + \frac{5}{3t^3} \right]_5^{\infty}$$

$$= \cancel{-}375 \left[\cancel{-}\left(\frac{1}{25} + \frac{\cancel{5}}{3 \times \underset{25}{\cancel{125}}} \right) \right]$$

$$= +375 \left(\frac{3+1}{75} \right)$$

$$= 5 \times 4$$

$$= 20$$