Probability & Statistics - 2

Assignment

(1)

$$P(Q) \rightarrow \frac{\hat{\theta} - \theta}{\sqrt{\hat{\lambda}/n}} \sim N(0,1)$$

$$\hat{\theta} = \frac{\Sigma x_i}{N} = 200$$

$C \cdot I = 95\%$

$$P\left(-1.96 < \frac{\hat{\theta} - \theta}{\sqrt{\hat{\lambda}/n}} < 1.96\right)$$

$$P\left(-1.96\sqrt{\frac{\hat{\lambda}}{n}} < \hat{\theta} - \theta < 1.96\sqrt{\frac{\hat{\lambda}}{n}}\right) = 0.9$$

$$P\left(\hat{\lambda} - 1.96\sqrt{\frac{\hat{\lambda}}{n}} < \theta < \hat{\theta} + 1.96\sqrt{\frac{\hat{\lambda}}{n}}\right) = 0.9$$

95% confidence interval for $\theta = (172.28, 227.72)$

Q2) i) $\bar{x} = \Sigma x_i / n$

$$E(X) = E(\bar{X}) = E\left(\frac{\Sigma x_i}{n}\right)$$

$$= \frac{1}{n} E(\Sigma x_i) = \frac{1}{n}\left[E(X_1) + E(X_2) + \dots + E(X_n)\right]$$

we know, $E(X) = \mu$.

$$\therefore \frac{1}{n}\left(\mu_1 + \mu_2 + \mu_3 + \dots \mu_n\right)$$

$$= \frac{n\mu}{n} = \mu.$$

(i) $v(\bar{x}) = v\left(\frac{\Sigma x_i}{n^2}\right)$

$= \frac{1}{n^2} v(\Sigma x_i)$

$= \cancel{\frac{1}{n^2} v} \; \frac{1}{n^2}\left[v(x_1) + v(x_2) + v(x_3) + \dots v(x_n)\right]$

we know, $v(x) = \sigma^2$.

$\therefore \frac{1}{n^2}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \dots \sigma_n^2\right)$

$= \frac{n\sigma^2}{n^2} \Rightarrow \frac{\sigma^2}{n}$.

(Hence Proved).

(ii) show that $E(s^2) = \sigma^2$

we know,

$$s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$E(s^2) = \frac{1}{n-1} E\left(\Sigma x_i^2 + \bar{x} - 2\Sigma x_i \bar{x}\right)$$

$$= \frac{1}{n-1}\left(E(\Sigma x_i^2) + \Sigma E(\bar{x})^2 - 2 \cdot \Sigma x_i \bar{x}\right)$$

$$\frac{1}{n-1}\left[nE(x_i^2)+nE(\bar{x})^2-2E(\bar{x}_1)E(\bar{x})\right]$$

$$=\frac{1}{n-1}\left[nE(xi^2)-E(\bar{x})^2\right]$$

$$E(x_1^2)=\left[v(x_1)+[E(x_1)]^2\right]$$

$$\sigma^2+\mu^2$$

$$E(\bar{x}^2)=V(\bar{x})+[E(x)]^2$$

$$=\frac{\sigma^2}{n}+\mu^2$$

$$\therefore \frac{1}{n-1}\left(n\sigma^2+\cancel{n\mu^2}-\sigma^2-\cancel{n\mu^2}\right)$$

$$\frac{n\sigma^2-\sigma^2}{n-1}$$

$$\frac{\sigma^2\cancel{(n-1)}}{\cancel{(n-1)}}=\sigma^2$$

(Hence proved)

(iii) show that,

$$var(s^2)=\frac{2\sigma^4}{(n-1)}$$

$$x_1\sim N(\mu,\sigma^2)$$

$$\frac{(n-1)s^2}{\sigma^2}\sim\chi^2_{n-1}$$

$$Var(s^2) = V\left(\frac{(n-1)s^2}{\sigma^2}\right) \frac{\times \sigma^4}{(n-1)^2}.$$

$$= V\left(\chi^2_{n-1}\right) \frac{\sigma^4}{(n-1)^2}.$$

$$= \frac{2\cancel{(n-1)}\sigma^4}{(n-1)^2} \Rightarrow \frac{2\sigma^4}{(n-1)}.$$

(Hence Proved)

(iv) given :-

$n = 10$

$\sigma^2 = 100$

$$P\left[-0.5E(s^2) < s^2 < 0.5E(s^2)\right]$$

$$P(-50 < s^2 < 50)$$

$$P\left(\frac{9 \times -50}{100} < \frac{(n-1)s^2}{\sigma^2} < \frac{9 \times +50}{100}\right)$$

$$P\left(-45 < \chi^2_9 < 45\right)$$

$$P(\chi^2_9 < 4.5) - P(\chi^2_9 \leq 4.5)$$

$$= 0.1245$$

Q5) i) $E(X) = \bar{x}$

$$\bar{x} = \frac{a+b}{2}$$

$$\therefore a = 2\bar{x} - b.$$

$$V(X) = s^2$$

$$\frac{(b-a)^2}{12} = s^2$$

$$b - a = 2\sqrt{3}s.$$

$$a = b - 2\sqrt{3}s$$

$$2\bar{x} - b = b - 2\sqrt{3}s$$

$$2b = 2\bar{x} + 2\sqrt{3}s$$

$$\therefore \hat{b} = \bar{x} + \sqrt{3}s.$$

$$a = 2\bar{x} - \bar{x} - \sqrt{3}s.$$

$$\hat{a} = \bar{x} - \sqrt{3}s.$$

(iv) $X \sim U(0, b)$

$$f(x) = \frac{1}{b-a} = \frac{1}{b}$$

$$L = f(x_1) \cdot f(x_2) \cdots f(x_n)$$

$$= \frac{1}{b} \cdot \frac{1}{b} \cdots = \frac{1}{b^n}$$

$$l = -n \log b.$$

$$\therefore \frac{dl}{db} = \frac{-n}{b} = 0$$

$$\frac{d^2 l}{db^2} = \frac{n}{b^2}$$

Q7) $\Sigma X_{pub} = 270$, $\bar{X}_{pub} = 30$.

$\Sigma X_{pri} = 315.5$, $\bar{X}_{pri} = 35.06$

$\Sigma X^2_{pub} = 8614.5$, $S_1^2 = \dfrac{8614.5 - 9 \times 30^2}{8}$

$$= 64.31$$

$$S_2^2 = \frac{11599.25 - 9 \times 35.06^2}{8} = 67.42$$

(ii) $H_0: \sigma_1^2 = \sigma_2^2$, $H_1: \sigma_1^2 \neq \sigma_2^2$.

$$TS = \cancel{s_1^2/s} \frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} \sim F_{n_1-1, n_2-1}$$

$$= \frac{64.31}{67.42} = 0.9544 \sim F_{8,8}$$

we reject Ho at L.S 5%.

(iii). $H_0: \mu_1 = \mu_2$, $H_1: \mu_1 < \mu_2$.

$$T.S = \frac{(\bar{X}_2 - \bar{X}_1) - (\mu_2 - \mu_1)}{\sqrt{s_p^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} \sim t_{n_1+n_2-2}.$$

$$\therefore s_p^2 = \frac{64.31 \times 8 + 67.42 \times 8}{16}$$

$$= 65.86$$

$$\therefore TS = \frac{(35.06 - 30) - 0}{\sqrt{65.86}\sqrt{\frac{1}{9} + \frac{1}{9}}}$$

$$= 1.32$$

we reject Ho @ 5% LS.

Q8) i) $\hat{\theta}$ is unbiased only and only if $E(\hat{\theta}) = \theta$.

ii) Measure of biasness is $E(\hat{\theta}) - \theta$

iii) MSE of estimator $\hat{\theta}$ = $(E(\hat{\theta}) - \theta)^2$

iv) –

v) An estimator is consistent if MSE converges to zero as sample size tends to ∞.

Q9) i) Variable t_k, $t_k = \frac{N(0,1)}{\sqrt{\chi^2_k / k}}$

where $k \to$ degree of freedom

(ii) Mean & Var of t_k for $k > 2$ are 0 & $\frac{k}{k-2}$ resp.

iii) $\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$

b) $t_{n-1} \sim N\left(0, \sqrt{\frac{n-1}{n-3}}\right) \sim N\left(0, \sqrt{9/7}\right)$.

$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim N(0, \sqrt{9}/\sqrt{7})$

CI for E $\left(50 - 2.58\sqrt{\frac{49.66}{10} \times \frac{9}{7}}, 50 + 2.58\sqrt{\frac{49.66}{10} \times \frac{9}{7}}\right)$

$= (43.54, 56.45)$

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Q10) i) $P(\text{Type 1 error}) = P(\text{Rej } H_0 / H_0 \text{ is T})$

$X \sim Poi(nu)$ ie. $Poi(3000)$

H_0, $X \sim Poi(3000)$

$P(\text{Reject } H_0 / H_0 \text{ is T}) = P(X > 3100 \text{ when } X \sim Poi\,3000)$

$X \sim N(3000, 3000)$

$$P(X > 3100) = P\left(Z > \frac{3100 - 300}{\sqrt{3000}}\right)$$

$1 - P(Z < 1.825) = 3.39\%$

ii) $P(\text{Type 2 error}) = P(\text{Accepting } H_0 / H_0 \text{ is F})$

iv) $\hat{u} \sim N(u, \hat{u}/n)$

$$\frac{\hat{u} - u}{\sqrt{\hat{u}/n}} \sim N(0,1)$$

$$P\left(-2.5758 < \frac{2.9 - H}{\sqrt{2.9/1000}} < 2.5758\right)$$

$= 0.99$

$u \in (2.7613, 3.0387)$

Q6) i)

Q6) i) $P(\text{type 1 error}) = P(\text{Reject } H_0 / H_0 \text{ is T})$

X be no. of hits in 1^{st} step 12 missiles
Y be no. of hits in 2nd step 12 missiles

$$P(\text{Type 1 error}) = P(X \geq 3/p=0.1) + P(X=0/p=0.1) \cdot P(Y \geq 5/p=0.1) + P(X=1/p=0.1) \cdot P(Y \geq 4/p=0.1) + P(X=2/p=0.1) \cdot P(Y \geq 3/p=0.1)$$

$$= (1-0.8891) + (0.2824)(1-0.9957) + (0.6590-0.2824)(1-0.9944) + (0.8891-0.6590)(1-0.08891)$$

$$= 0.14727 \approx 15\%$$

(ii) $P(X \geq 3/p=0.3) + P(X=0/p=0.3) \cdot P(Y \geq 5, p=0.3) + P(X=1/p=0.3) \cdot P(Y \geq 4/p=0.3) + P(X=2/p=0.3) \cdot P(Y \geq 3/p=0.3)$

$$= 0.91253 \approx 91\%$$

(iii) $P(\text{Type II error})$ is the prob. of accepting H_0 when H_0 is f.

$$= 1 - 0.91253$$
$$= 0.08747 \approx 9\% \ (p=0.3)$$

iv) $X \sim Bin(120, 1/3)$ $\qquad \hat{p} = \frac{40}{120} = \frac{1}{3}$

$$\frac{X/n - p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$$

$$P\left(-1.2816 < \frac{\hat{p}-p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} < 1.2816\right)$$

At CI 90% $p \in (0.333, 0.2782)$

v) $p > 0.278$ @ 10% LS

$H_0: p = 0.278$
$H_1: p > 0.278$

$$\frac{X - np_0}{\sqrt{np_0q_0}} \sim N(0,1)$$

$$\frac{39.5 - 120 \times 0.278}{\sqrt{120 \times 0.278 \times 0.722}}$$

$= 1.2511$