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Assignment 2

Subject – SRM-4

Q1)

i.

$Y_t^2 - \beta_1 e_t^2 Y_{t-1}^2 = 2(Y_t - \beta_1 e_t^2 Y_{t-1})\mu - (1 - \beta_1 e_t^2)\mu^2 + \beta_0 e_t^2$

$Y_t^2 - 2Y_t\mu + \mu^2 = e_t^2(\beta_0 + \beta_1 Y_{t-1}^2 - 2\beta\_1Y(t-1)\mu + \beta_1\mu^2)$

$\therefore (Y_t - \mu)^2 = e_t^2\left(\beta_0 + \beta_1\left(Y_{(t-1)} - \mu\right)^2\right)$

$\therefore Y_t = \mu + e_t\left(\beta_0 + \beta_1\left(Y_{(t-1)} - \mu\right)^2\right)^{0.5}$

$E[Y_t] = E[\mu] + E(e_t\left(\beta_0 + \beta_1\left(Y_{(t-1)} - \mu\right)^2\right)^{0.5}$

$e_t and\ Y_{t-1} are\ independent$

$\therefore E[Y_t] = \mu + 0 * E\left(\left(\beta_0 + \beta_1\left(Y_{(t-1)} - \mu\right)^2\right)^{0.5}\right)$

$\therefore E[Y_t] = \mu$

$Cov(Y_t, Y_{t-1}) = E(Y_tY_{t-s}) - E(Y_t)E(Y_{t-s})$

$$\therefore Cov(Y_t, Y_{t-1}) = \mu^2 + \mu E(e_t)E\left(\left(\beta_0 + \beta_1\left(Y_{(t-1)} - \mu\right)^2\right)^{0.5}\right) + \mu E(e_{t-s})E\left(\left(\beta_0 + \beta_1\left(Y_{(t-s-1)} - \mu\right)^2\right)^{0.5}\right) + E(e_t)E(e_{t-s})E\left(\left(\left(\beta_0 + \beta_1\left(Y_{(t-1)} - \mu\right)^2\right)^{0.5}\right)\left(\left(\beta_0 + \beta_1\left(Y_{(t-s-1)} - \mu\right)^2\right)^{0.5}\right)\right) - \mu^2$$

$\therefore Cov(Y_t, Y_{t-1}) = \mu^2 - \mu^2$

$\therefore Cov(Y_t, Y_{t-1}) = 0$

ii.

$Var(Y_t|Y_{t-1}) = Var(e_t)Var\left(\beta_0 + \beta_1\left(Y_{(t-1)} - \mu\right)^2\right) = \beta_0 + \beta_1\left(Y_{(t-1)} - \mu\right)^2$

$$\therefore var\ of\ Y_t is\ dependent\ on\ Y_{t-1}.\ var\ of\ Y_t\ will\ depend\ on\ Y_{t-s}.$$
$$\therefore Y_t\ and\ Y_{t-s}\ are\ dependent$$

iii.

$\Delta X_t = X_t - X_{t-1}$

$E(\Delta X_t) = E(X_t) - E(X_{t-1})$

$\therefore E(\Delta X_t) = E(0.5Y_t + 0.3t + 0.1) - E(0.5Y_{(t-1)} + 0.3(t-1) + 0.1$

$\therefore E(\Delta X_t) = 0.5\,\mu + 0.3t + 0.1 - 0.5\mu - 0.3(t-1) - 0.1 = 0.3$

$\therefore Constant$

$Cov(\Delta X_t, \Delta X_{t-s}) = Cov(X_t - X_{t-1}, X_{t-s} - X_{t-s-1})$

$= Cov(0.3 + Y_t - Y_{t-1}, 0.3 + Y_{t-s} - Y_{t-s-1})$

$= Cov(Y_t - Y_{t-1}, Y_{t-s} - Y_{t-s-1})$

$= Cov(Y_t - Y_{t-s}) - cov(Y_t - Y_{t-s-1}) - cov(Y_{t-1} - Y_{t-s}) + cov(Y_{t-1}, Y_{t-s-1})$

$= 0$

$\therefore Auto\ correlation\ function\ is\ constant\ and\ sp\ the\ first\ difference\ of\ X_t is\ stationary$

Q2

i.

The characteristic equation is: 1-z-0.5z$^2$+0.5z$^2$=0

Rewriting in terms of X=(1-B)Y, we get

$X_t - 0.5X_{t-2} = Z_t + 0.3Z_{t-1}, which\ is\ ARIMA(2,1), \therefore Y\ is\ ARIMA(2,1,1)$

ii. $The\ charateristic\ polynomial\ of\ X\ is\ (1 - 0.5z^2) whose\ roots\ are\ \pm \sqrt{2}. As\ both\ roots\ are > 1, the\ process\ is\ stationary$

iii. The model equation is: $X_t = 0.5X_{t-2} + Z_t + 0.3Z_{t-1}$

$Cov(X_t, Z_t) = 0 + \sigma^2 + 0 = \sigma^2$

$Cov(X_t, Z_{t-1}) = 0 + 0 + 0.3\sigma^2 = .3\,\sigma^2$

$Cov(X_t, Z_{t-2}) = 0 + 0 + 0.5\sigma^2 = .5\,\sigma^2$

$\gamma(0) = cov(X_t, X_t)$

$\quad = 0.5\gamma(2) + \sigma^2 + 0.09\sigma^2 = \ 0.5\gamma(2) + 1.09\sigma^2 ----(1)$

$\gamma(1) = cov(X_t, X_{t-1}) = 0.5\gamma(1) + 0 + 0.3\sigma^2 --(2)$

$\gamma(2) = cov(X_t, X_{t-2}) = 0.5\gamma(0) --(3)$

$\gamma(k) = cov(X_t, X_{t-k}) = 0.5\gamma(k-2); k > 2 --- (4)$

$By\ substituting\ (3) in\ (1), we\ get, \gamma(0) = 0.25\gamma(0) + 1.09\sigma^2, i.e. \gamma(0) = \dfrac{109\sigma^2}{75}$

$From(2)\ \gamma(1) = \dfrac{3\sigma^2}{5}$

$$\therefore \rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{45}{109}$$

$From\ (3) and\ (4), \rho(k) = 0.5\rho(k-2) for\ k \geq 2$

$\rho(k) = 0.5^{\frac{|k|}{2}}\ if\ |k| is\ even$

$= \left(\frac{45}{109}\right)(0.5)^{(|k|-1)/2}\ if\ |k| is\ odd$

Q3

i.

a. $The\ process\ is\ (1 - 0.4B - 0.2B^2)Y_t = (1 + 0.025B)Z_t + 0.016$
$Characteristic\ eq\ is\ 1 - 0.4z - 0.2z^2 = 0$
$There\ is\ no\ root\ having\ magnitude\ 1. \therefore d = 0$
∴it is ARIMA(2,0,1) process

b. $(1 - 0.4 - 0.2)E(Y_t) = 0.016$
$$\therefore E(Y_t) = \frac{0.016}{0.4} = 4\%$$

c. 2 roots are -1±√6 i.e. 1.4495 and -3.4495 both of which are >1 when mod is taken
∴$Y_t$ is a stationary process

ii.

a.

b. Inspection of the graph of time-plot of residuals: On observing the graph, one could see a pattern such as clusters or uneven fluctuations which indicate inadequate fit.
Inspection of sample autocorrelation function of residuals: If too many residual values lie beyond the range ±2/√N indicates poor fit or too few parameters.

Q4

i. AR(2) process
$Model\ can\ be\ written\ as: (1 + \alpha B - \alpha^2 B^2)Y_t = Z_t$
$\therefore Characteristic\ eq\ is: 1 + \alpha x - \alpha^2 x^2 = 0$
Using quadratic formula
$$x = \frac{1 \pm \sqrt{5}}{2\alpha}$$
$$We\ want\ |\alpha| > 1\ \therefore \frac{\sqrt{5}-1}{2|\alpha|} > 1; \frac{\sqrt{5}+1}{2|\alpha|} > 1$$
$$\therefore |\alpha| < \frac{\sqrt{5}-1}{2}$$

ii. $Y_t = -\alpha Y_{t-1} + \alpha^2 Y_{t-2} + Z_t$
Yule-Walker equations are:

$Cov[Y_t, Y_t] = \gamma_0 = -\alpha.\gamma_1 + \alpha^2\gamma_2 + \sigma^2 ----Eq(1)$
$Cov[Y_t, Y_{t-1}] = \gamma_1 = -\alpha.\gamma_0 + \alpha^2\gamma_1 ----- Eq(2)$

$$Cov[Y_t, Y_{t-2}] = \gamma_2 = -\alpha.\gamma_1 + \alpha^2\gamma_0 - - - - - Eq(3)$$
$$From\ eq(2); \gamma_1 = -\frac{\alpha\gamma_0}{1-\alpha^2} - - - - - Eq(4)$$
$$Substituting\ (4) in\ (3);$$
$$\gamma_2 = -\alpha\left[-\frac{\alpha\gamma_0}{1-\alpha^2}\right] + \alpha^2\gamma_0 = (2\alpha^2 - \alpha^4).\frac{\gamma_0}{1-\alpha^2} - - - Eq(5)$$
$$\therefore \gamma_0 = \frac{\sigma^2(1-\alpha^2)}{1-2\alpha^2-2\alpha^4+\alpha^6}$$
$$\therefore \gamma_1 = \frac{\sigma^2(-\alpha^1)}{1-2\alpha^2-2\alpha^4+\alpha^6}$$
$$\therefore \gamma_2 = \frac{\sigma^2(2\alpha^2-\alpha^4)}{1-2\alpha^2-2\alpha^4+\alpha^6}$$

Q5

i. The model is

$$(X_n - X_{n-1}) = \alpha(X_{n-1} - X_{n-2}) + \epsilon_n$$
$$\alpha = \rho_1 = \sum_{i=3}^{200}(X_i - X_{i-1})(X_{i-1} - X_{i-2}) / \sum_{i=2}^{200}(X_i - X_{i-1})^2$$
$$\therefore \alpha = \frac{587.83}{936.49} = 0.6277$$
$$\gamma_0 = \frac{1}{200}\sum_{i=2}^{200}(X_i - X_{i-1})^2 = \frac{936.49}{200} = 4.6825$$
$$using\ \gamma_0 = \frac{\sigma^2}{1-\rho_1^2} = \frac{\sigma^2}{1-\alpha^2}, we\ get$$
$$\sigma_{hat}^2 = (1-\alpha_{hat}^2)\gamma_{0_{hat}} = (1-0.6277^2) * 4.6825 = 2.8376$$

ii. Forecast of $x_{201}$ is:

$$\left(x_{200_{hat}}(1) - x_{200}\right) = \alpha_{hat}(x_{200} - x_{199}) = 0.6277(1.93 - 0.82) = 0.6967$$
$$\therefore x_{200_{hat}}(1) = x_{200} + 0.6967 = 1.93 + 0.6967 = 2.6267$$

Q6

i. The purpose of a practical time series analysis may be summarized as follows:

- Description of the data
- Construction of a model which fits the data
- Forecasting future values of the process
- For vector time series, investigating connections between two or more observed processes with the aim of using values of some of the processes to predict those of the others
- Deciding whether the process is out of control, requiring action

ii. AR(1) and random walk are Markov

MA(1) and AR(2) are not Markov

iii.

a. E[X(t)]=a+bt+E[Y(t)]
Since Y(t) is stationary, E[Y(t)]=c that does not depend on t
∴ E[X(t)]=a+bt+c, which depends on t
It follows that X(t) cannot be stationary

b. E[∇ X(t)]= E[X(t)]- E[X(t-1)]=a+bt+c-[a+b(t-1)+c]=b
Thus, the mean does not depend on t
$Cov[\ \nabla X(t), \nabla X(s)] = Cov[\{X(t) - X(t-1)\}, \{X(s) - X(s-1)\}]$

$$= Cov[\{b + Y(t) - Y(t-1)\}, b + Y(s) - Y(s-1)\}]$$

$$= Cov[\ \{Y(t) - Y(t-1)\}, Y(s) - Y(s-1)\}]$$

$$= Cov[Y(t), Y(s)] - Cov[Y(t), Y(s-1)] - Cov[Y(t-1), Y(s)] + Cov[Y(t-1), Y(s-1)]$$

$$= C_{Y(t-s)} - C_Y(t-s+1) - C_Y(t-s-1) + C_Y(t-s)$$

$Last\ expression\ depends\ on\ t\ and\ s\ only\ through\ the\ differnce\ t-s.$

$\therefore \nabla X(t) is\ stationary$

$As\ \nabla X(t)\ is\ stationary\ but\ X(t) is\ not\ the\ process, the\ process\ X(t)\ is\ I(1)$

iv. $X_1(t) - X_2(t) = a[X_1(t-1) - X_2(t-1)] + b[X_2(t-1) - X_1(t-1)] + [e_1(t) - e_2(t)]$

$$= (a-b)[X_1(t-1) - X_2(t-1)] + [e_1(t) - e_2(t)]$$

$Let\ Y(t) = X_1(t) - X_2(t) and\ e(t) = e_1(t) - e_2(t)$

$Since\ e_1(t)\ and\ e_2(t)\ are\ WNP, \therefore\ e(t) is\ also\ WNP$

$\therefore\ Y(t) must\ be\ an\ AR(1) process$

$Y(t) is\ stationary\ as\ long\ as\ |a-b| < 1$

Q7

i.

a. Auto Regressive process (AR)
An auto regressive process with order p is a sequence of random variables ($X_t$) defined consecutively by the rule:

$$X_t = \mu + \alpha_1(X_{t-1} - \mu) + \alpha_2(X_{t-2} - \mu) + \cdots + \alpha_p\left(X_{t-p} - \mu\right) + e_t$$

Thus the AR(p) model attempts to explain the current value of X as a linear combination of past values with some additional externality generated random variable

b. Moving average process (MA)

A moving average process with order q is a sequence defined by:

$X_t = \mu + e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \cdots + \beta_q e_{t-q}$

The MA(q) model explains the relation between $X_t$ as an indirect effect, arising from the fact that the current value of the process results from the recently past random errors as well as the current one.

c. Autoregressive moving average process (ARMA)

The two basic processes (AR and MA) can be combined to give an auto regressive moving average or ARMA process. The eq$^n$ is :

$$X_t = \mu + \alpha_1(X_{t-1} - \mu) + \alpha_2(X_{t-2} - \mu) + \cdots + \alpha_p\left(X_{t-p} - \mu\right) + e_t + e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \cdots + \beta_q e_{t-q}$$

ii.

a. It is MA(1) process and hence it is stationary . Also, it is ARIMA(0,0,1)

b. It is an ARMA(2,3) process.

This process cannot be differenced, so to classify it as ARIMA(2,0,3), we must check that it is I(0) i.e. stationary

$\because (1 - 1.4B^2)X_t = e_t + 0.5e_t$

$\phi(\lambda) = 1 - 1.4\lambda^2 = 0$

$\therefore \lambda = \pm 0.8452$

$\because both\ the\ roots\ are < 1, \therefore I\ is\ a\ non-stationary\ ARMA(2,3) process$

c. It is an ARMA(2,1) process

$$X_t - 1.4X_{t-1} + 0.4X_{t-2} = e_t + e_{t-1}$$

$$(X_t - X_{t-1}) - 0.4(X_{t-1} - X_{t-2}) = e_t + e_{t-1}$$

$$\therefore \nabla X_t - 0.4\nabla X_{t-1} = e_t + e_{t-1}$$

$$This\ cannot\ be\ differenced\ again, so\ to\ classify\ it\ as\ ARIMA(1,1,1) we\ must\ verify\ whether\ this\ differenced\ process\ is\ stationary\ \left(i.e. the\ original\ process\ is\ I(1)\right).$$

$$\therefore It\ is\ an\ ARIMA(1,1,1)$$

iii. The process is stationary as it is the sum of stationary white noise terms

The Autocorrelation function is:

$\gamma_0 = Cov(X_t, X_t) = var(X_t)$

$= Cov(e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3}, e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3})$

$= \sigma^2 + 0.25^2\sigma^2 + 0.5^2\sigma^2 + 0.25^2\sigma^2 = 1.375\sigma^2$

$\gamma_1 = Cov(X_t, X_{t-1})$

$= Cov(e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3}, e_{t-1} + 0.25e_{t-2} + 0.5e_{t-3} + 0.25e_{t-4})$

$= 0.25\sigma^2 + 0.5 * 0.25 * \sigma^2 + 0.5 * 0.25 * \sigma^2 = 0.5\sigma^2$

$\gamma_{\pm 2} = Cov(X_t, X_{t-2})$

$= Cov((e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3}, e_{t-2} + 0.25e_{t-3} + 0.5e_{t-4} + 0.25e_{t-5})$

$$= 0.5\sigma^2 + 0.25^2\sigma^2 = 0.5625\,\sigma^2$$

$\gamma \pm 3 = Cov(X_t, X_{t-3})$

$= Cov((e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3}, e_{t-3} + 0.25e_{t-4} +$
$0.5e_{t-5} +$ $0.25e_{t-6}) = 0.25\sigma^2$

$$\gamma_{\pm 0} = 0\ for\ |k| > 3$$

$$\rho_0 = 1$$
$$\rho_{\pm 1} = 0.364$$
$$\rho_{\pm 2} = 0.409$$
$$\rho_{\pm 3} = 0.182$$
$$\rho_{\pm k} = 0\ for\ |k| > 3$$

Q8

i. $X_t = A_t + B_t$

$A_t = 0.5A_{t-1} + 0.5B_{(t-1)} + e_1^{(A)}$

$B_t = 0.7A_{t-1} - 0.7A_{(t-2)} + e_1^{(B)}$

Using matrix notation we get,

$$Y_t = \begin{pmatrix} A_t \\ B_t \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.7 & 0 \end{pmatrix} \begin{pmatrix} A_{t-1} \\ B_{t-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -0.7 & 0 \end{pmatrix} \begin{pmatrix} A_{t-2} \\ B_{t-2} \end{pmatrix} + \begin{pmatrix} e_1^{(A)} \\ e_1^{(B)} \end{pmatrix}$$

$$Y_t = \begin{pmatrix} A_t \\ B_t \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.7 & 0 \end{pmatrix} Y_{t-1} + \begin{pmatrix} 0 & 0 \\ -0.7 & 0 \end{pmatrix} Y_{t-2} + \begin{pmatrix} e_1^{(A)} \\ e_1^{(B)} \end{pmatrix}$$

The Eigen values of first matrix are given by the following equation,

$\lambda^2 - 0.5\lambda - 0.35 = 0$

Hence, $|\lambda| < 1$

Similarly for second matrix, $|\lambda| < 1$

As all the Eigen values are less than 1, hence the process $Y_t$ is stationary.

ii. (a)

$X_t = (\alpha + 1)X_{t-1} - (\alpha + 0.25\alpha^2)X_{t-2} + 0.25\alpha^2 X_{t-3} + e_t$

$X_t - X_{t-1} = \alpha(X_{t-1} - X_{t-2}) - 0.25\alpha^2(X_{t-2} - X_{t-3}) + e_t$

$Y_t = \alpha Y_{t-1} - 0.25\alpha^2 Y_{t-2} + e_t, \quad assuming\ Y_t = X_t - X_{t-1}$

$So\ X_t\ is\ ARIMA(2,1,0)\ process\ if\ it\ is\ I(1)$

Now, $(1 - \alpha\beta + 0.25\alpha^2\beta^2)Y_t = e_t$

The characteristic equation is, $1 - \alpha\lambda + 0.25\alpha^2\lambda^2 = 0 \quad \therefore \lambda = \frac{2}{\alpha}$

To meet the stationary condition, $|\lambda| > 1,\ \ |\alpha|\ should\ be\ less\ then\ 2$
Hence $X_t\ is\ ARIMA(2,1,0)\ process\ with\ |\alpha| < 2$
(b)
Now $Cov(Y_t, e_t) = Cov(e_t, e_t) = \sigma^2$
$\gamma_0 = Cov(Y_t, Y_t) = \alpha\gamma_1 - 0.25\alpha^2\gamma_2 + \sigma^2 \quad \ldots(1)$
Taking co-variance with $Y_{t-1}, Y_{t-2}\ and\ Y_{t-k}\ we\ get,$
$\gamma_1 = \alpha\gamma_0 - 0.25\alpha^2\gamma_1 \ldots\ldots(2)$
$\gamma_2 = \alpha\gamma_1 - 0.25\alpha^2\gamma_0 \ldots\ldots(3)$
$\gamma_k = \alpha\gamma_{k-1} - 0.25\alpha^2\gamma_{k-2}$
From (2), $\gamma_1 = \frac{\alpha\gamma_0}{1+0.25\alpha^2}$ ……..(4)
Substituting (3) in (1) we get,
$\gamma_0 = \alpha\gamma_1 - 0.25\alpha^2(\alpha\gamma_1 - 0.25\alpha^2\gamma_0) + \sigma^2$

$$\therefore (1-(025\alpha^2)^2)\gamma_0 = \frac{\alpha^2(1-0.25\alpha^2)}{1+0.25\alpha^2}\gamma_0 + \sigma^2$$

$$\therefore \gamma_0 = \frac{1+0.25\alpha^2}{(1-0.25\alpha^2)^3}\sigma^2$$

$$Hence, \gamma_1 = \frac{\alpha}{1+0.25\alpha^2}\left(\frac{1+0.25\alpha^2}{(1-0.25\alpha^2)^3}\right)\sigma^2 = \frac{\alpha}{(1-0.25\alpha^2)^3}\sigma^2$$

$For\ k \geq 2, \gamma_k = \alpha\gamma_{k-1} - 0.25\alpha^2\gamma_{k-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots.(5)$
$Now\ \gamma_k follows\ the\ below\ eq^n$
$\lambda_1 + k\lambda_2 = (0.5\alpha)^{-k}\gamma_k \ \ldots(6)$
$Substituting\ the\ above\ two\ eq^n in\ eq^n(5), we\ get$

$$\gamma_k = \alpha(0.5\alpha)^{k-1}[\lambda_1 + (k-1)\lambda_2] - 0.25\alpha^2(0.5\alpha)^{k-2}[\lambda_1 + (k-2)\lambda_2]$$

$$\therefore \gamma_k = \lambda_1\alpha^k(0.5)^{k-1}\left[1 - \frac{0.25}{0.5}\right] + \lambda_2\alpha^k(0.5)^{k-1}\left[(k-1) - \frac{1}{2}(k-2)\right]$$

$$\therefore \gamma_k = \lambda_1\alpha^k(0.5)^k = k\lambda_2\alpha^k(0.5)^k$$

$$\therefore \lambda_1 + k\ \lambda_2 = (0.5\alpha)^{-k}\gamma_k$$

$which\ is\ the\ original\ eq^n. Hence\ \gamma_k follows\ eq^n(6)$
Now Putting k = 0, we get

$$\lambda_1 = \gamma_0 = \frac{(1+0.25\alpha^2)}{(1-0.25\alpha^2)^3}\sigma^2$$

$Putting\ k = 1, we\ get$

$$\lambda_2 = \frac{1}{0.5\alpha}\gamma_1 - \ \lambda_1 = \frac{\sigma}{(1-0.25\alpha^2)^2}$$

$(c)$
$\alpha = 0.04, hence\ Y_t = 0.04Y_{t-1} - 0.0004Y_{t-2} + e_t$
$Y_t = X_1 - X_{t-1}\ \ OR\ X_t = Y_t + X_{t-1}$
$Since\ x_1, x_2, \ldots, x_{50}\ are\ observed\ values$
$X_{51} = Y_{51} + X_{50}$
$X_{52} = Y_{52} + X_{51}$
$So\ the\ forcasted\ values\ are$
$x_{51} = y_{51} + x_{50}\ and\ \ x_{52} = y_{52} + x_{51}$
$Where$
$y_{51} = 0.04(x_{50} - x_{49}) - 0.0004(x_{49} - x_{48})$
$And\ y_{52} = 0.04y_{51} - 0.0004(x_{50} - x_{49})$

Q10

i. $p = 1, q = 4, hence\ it\ will\ follow\ ARMA(1,4)$

ii. $Non - linear\ non\ stationary\ time\ series\ models\ includes: -$

$Bilinear\ models\ are\ those\ that\ exhibit$ bursty $behaviour:$

$X_n - \alpha(X_{n-1} - \mu) = \mu + e_n + \beta e_{n-1} + b(X_{n-1} - \mu)e_{n-1}$

$Threshold\ autoregressive\ models\ are\ used\ to\ model$ cyclical $behaviour:$

$X_n = \mu + \alpha_1(X_{n-1} - \mu) + e_n \dots, if\ X_{n-1} \leq d,$

$X_n = \mu + \alpha_2(X_{n-1} - \mu) + e_n \dots, if\ X_{n-1} \leq d,$

$Random\ coeffient, autoregressive\ models\ is\ a\ sequence\ of$
$independent\ random\ variables:$

$X_t = \mu + \alpha(X_{t-1} - \mu) + e_t\ , where\ \{\alpha_1, \alpha_2, \dots, \alpha_n\}$
$is\ a\ sequence\ of\ independent$
$random\ variables.$

$Autoregressive\ with\ conditional\ hetreoscedasticity\ (ARCH)\ models$
$are\ used\ to\ model\ asset\ prices, where\ we\ require\ the\ volatility\ to\ depend\ on$
$the\ size\ of\ the\ previous\ value:$

$$X_t = \mu + e_t\sqrt{\alpha_0 + \sum_{k-1}^{p} \alpha_k(X_{t-k} - \mu)^2}$$

iii. $X_t\ follows\ MA, hence\ we\ can\ write$

$X_t = e_t + \beta e_{t-1}\ \ , where\ e_t \sim (0, \sigma^2)$

$Now, var(X_t) = var(e_t + \beta e_{t-1})$

$= var(e_t) + \beta^2 var(e_{t-1})$

$= (1 + \beta^2)\sigma^2 \qquad \dots (1)$

$Now, \Delta Y_t = (0.6 + 0.3t + X_t) - [0.6 + 0.3(t - 1) + X_{t-1}]$

$= 0.3 + (X_t - X_{t-1})$

$Hence, var(\Delta Y_t) = [cov(X_t - X_{t-1}, X_t - X_{t-1})]$

$= [2\gamma_X(0) - \gamma_X(-1) - \gamma_X(1)] \dots (2)$

$\therefore Now, \gamma_x(0) = (1 + \beta^2)\sigma^2$

$And, \gamma_x(1) = \gamma_x(-1) = cov(e_t + \beta e_{t-1}, e_t + \beta e_{t-1}) = \beta\sigma^2$

$\therefore from\ (2) we\ get,$

$var(\Delta Y_t) = [2(1 + \beta^2)\sigma^2 - 2\beta\sigma^2] = 2\sigma^2[1 - \beta + \beta^2]$

$Now, var(\Delta Y_t) - var(X_t)$

$= [2 - 2\beta + 2\beta^2]\sigma^2 - (1 + \beta^2)\sigma^2$

$= [1 - 2\beta + \beta^2]\sigma^2$

$= (1 - \beta^2)\sigma^2 > 0$

$Hence\ the\ standard\ deviation\ of\ first\ difference\ of\ Y_t\ is\ higher$
$than\ that\ of\ X_t$

Q11

i. Purposes of a practical time series analysis:

- description of the data
- construction of a model which fits the data
- forecasting future values of the process
- deciding whether the process is out of control, requiring action
- for vector time series, investigating connections between two or more observed processes with the aim of using values of some of the processes to predict those of the others

ii. Univariate time series

A univariate time series is a sequence of observations $\{X_t\}$ recorded at regular intervals. The state space is continuous, but the time set is discrete. Such series may follow a pattern to some extent, for example possessing a trend or seasonal component, as well as having random factors.

Invertibility

A time series process X is invertible if we can write the white noise term et as a convergent sum of the X terms. It can be shown that this is equivalent to saying that the roots of the characteristic polynomial of the e terms are all greater than 1 in magnitude. Invertibility is a desirable characteristic since it enables us to calculate the residual terms and hence analyse the goodness of fit of a particular model.

Markov

A time series process X has the Markov property if:

$P[X_t = a|X_{s1} = x_1, \ldots\ , X_{sn} = x_n\ , X_s = x] = P[X_t = a|X_s = x]$

$for\ all\ times\ s_1 < s_2 < \cdots < s_n < s < t\ and\ all\ states\ a, x_1, \ldots, x_n\ of\ S.$

$In\ other\ words\ we\ can\ predict\ the\ future\ state\ (\ at\ time\ t)\ from\ the\ current\ state\ (\ at\ time\ s)\ alone.$

iii. Following two methods can be used to remove the seasonal variation

1. Seasonal differencing

   Half yearly variations mean the period is two half years i.e. $h_t = h_{t+2}$

   Hence, we can subtract the value from 2 half years age:

   $\nabla_2 S_t = S_t - S_{t-2}$
2. Method of moving average

   Half yearly variation mean the period is two half years i.e. $h_t = h_{t+2}$

   We can find a symmetrical average of 2 terms about $S_t$

   $\frac{1}{2}\left(\frac{1}{2}S_{t-1} + S_t + \frac{1}{2}S_{t+1}\right)$

Following two methods can be used to remove any linear trend

1. Least square trend removal

   We can estimate $\alpha\ and\ \beta$ using least square regression method,

   Determine $\alpha\ and\ \beta$ which minimizes $\sum x_t^2 = \sum(y_t^2 - \alpha - \beta t)^2$

   Subtract the regression line from the observed values $S_t - \alpha - \beta t$
2. Differencing

   Subtract the previous observed values

   $\nabla S_t = S_t - S_{t-1}$

iv. a.

$Y_t - 0.6Y_{t-1} = 0.5 + 0.2e_t + 0.7e_{t-1} + 0.3e_{t-2}$

$\therefore (1 - 0.6B)Y_t = 0.5 + 0.2e_t + 0.7e_{t-1} + 0.3e_{t-2}$

$\therefore Y_t = (1 - 0.6B)^{-1}(0.5 + 0.2e_t + 0.7e_{t-1} + 0.3e_{t-2})$

$$\therefore Y_t = \sum_{0}^{\infty} 0.6^i B^i (0.5 + 0.2e_t + 0.7e_{t-1} + 0.3e_{t-2})$$

$$= \frac{0.5}{1 - 0.6} + 0.2\sum_{-}^{\infty} 0.6^i e_{t-i} + 0.7\sum_{-}^{\infty} 0.6^i e_{t-1-i} + 0.3\sum_{-}^{\infty} 0.6^i e_{t-2-i}$$

$$= 1.25 + 0.2(1e_t + 0.6e_{t-1}) + 0.7e_{t-1} + (0.3 + 0.7 \times 0.6 + 0.2 \times 0.6^2)\sum_{i=2}^{\infty} 0.6^{i-2} e_{t-i}$$

$$= 1.25 + 0.2e_t + 0.19e_{t-1} + 1.192\sum_{i=2}^{\infty} 0.6^{i-2} e_{t-i}$$

b.

Now

$Cov(Y_t, e_t) = Cov(0.6Y_{t-1} + 0.5 + 0.2e_t + 0.7e_{t-1} + 0.3e_{t-2}, e_t) = 0.2\sigma^2$

$Cov(Y_t, e_{t-1}) = Cov(0.6Y_{t-1} + 0.5 + 0.2e_t + 0.7e_{t-1} + 0.3e_{t-2}, e_{t-1})$
$= 0.82\sigma^2$

$Cov(Y_t, e_{t-2}) = Cov(0.6Y_{t-1} + 0.5 + 0.2e_t + 0.7e_{t-1} + 0.3e_{t-2}, e_{t-2})$
$= 0.792\sigma^2$

$Cov(Y_{t-1}, e_{t-2}) = Cov(0.6Y_{t-2} + 0.5 + 0.2e_{t-1} + 0.7e_{t-2} + 0.3e_{t-3}, e_{t-2})$
$= 0.82\sigma^2$

Now, $\gamma_0 = Cov(Y_t, Y_t) = Cov(Y_t, 0.6Y_{t-1} + 0.5 + 0.2e_t + 0.7e_{t-1} + 0.3e_{t-2})$
$= 0.6\gamma_1 + 0.2 \times 0.2\sigma^2 + 0.7 \times 0.82\sigma^2 + 0.3 \times 0.792\sigma^2$
$= 0.6\gamma_1 + 0.8516\sigma^2 \ldots (a)$

$\gamma_1 = Cov(Y_t, Y_{t-1}) = Cov(Y_{t-1}, 0.6Y_{t-1} + 0.5 + 0.2e_t + 0.7e_{t-1} + 0.3e_{t-2})$
$= 0.6\gamma_0 + 0.7 \times 0.2\sigma^2 + 0.3 \times 0.82\sigma^2 = 0.6\gamma_0 + 0.386\sigma^2 \ldots (b)$

$\gamma_2 = Cov(Y_1, Y_{t-2}) = Cov(Y_{t-2}, 0.6Y_{t-1} + 0.5 + 0.2e_t + 0.7e_{t-1} + 0.3e_{t-2})$
$= 0.6Y_t + 0.06\sigma^2$

$Now\ from\ (a)\ and\ (b),$

$\gamma_0 = 1.6925\sigma^2$

$\gamma_1 = 1.4015\sigma^2$

$\gamma_2 = 0.9009\sigma^2$

$and\ \gamma_k = 0.6\gamma_{k-1}, for\ k \geq 3$

Hence, $\rho_0 = 1, \rho_1 = 0.8281,\ \rho_2 = 0.5323,\ \rho_k = 0.6\rho_{k-1}\ for\ k \geq 3$

Q 12

i. Lack of stationarity:

- Lack of stationarity may be caused by the presence of deterministic effects in the quantity being observed. For ex. Deterministic trend or cycle such as seasonal effects.
- If the process observed is the integrated version of a more fundamental process.
- For ex. A company which sells greeting cards will find that the sales in some months will be much higher than in others.

ii. a.

Any of the below is a form of ARIMA(p,d,q) process

$$(1-B)^d\phi(B)(X_t-\mu)=\theta(B)e_t \; where \; \phi(B)=1-\sum_1^p B^i\alpha i \; and \; \theta(B) = \sum_1^q B^j\beta j$$

$$(\nabla^d X_t-\mu)=\sum_1^p \alpha i(\nabla^d X_t-1-\mu)+e_t+\sum_1^q B^j e_{t-j}$$

b.

Main steps involved in Box-Jenkins methodology

- Tentative identification of a model from the ARIMA class
- Estimation of the parameter in the identified model
- Diagnostic checks

c.

If the sample auto correlation coefficients decay slowly from 1 then this indicates that further differencing is required. This is not the case for d=1, which means that the differencing of the original series is required once hence d=1.
Further the correct value of d minimises the sample variance. This also indicates that d=1.

d.
Classify the time series as ARIMA(p,d,q)
1 $X_t=0.8e_{t-1}+e_t$
This is a MA(1) process and hence it is stationary. Therefore we can classify it as ARIMA(0,0,1) process.

2 $X_t=2X_{t-2}+e_t+0.5e_{t-3}$
This is an ARMA(2,3) process. This process cannot be differenced so we can classify it as ARIMA (2,0,3), we also must see if I(0) is stationary.

$=(1-2B^2)X_t=e_t+0.5e_{t-3}$, the characteristic equation of AR terms is

$=\phi(\lambda)=1-2\lambda^2=0$

$\lambda = \pm\frac{1}{\sqrt{2}}$, both the roots of the characteristic equation are less than 1. We can not classify the process as ARIMA(2,0,3). Hence it's a non-stationary ARMA(2,3) process.

3 $X_t = 1.5X_{t-1} + 0.2X_{t-2} + e_t + e_{t-1}$

This is an ARMA(2,1) process.

The process can be differenced as follows

$X_t - 1.5X_{t-1} - 0.2X_{t-2} = e_t + e_{t-1}$

$= (X_t - X_{t-1})$

(not able to solve further)

Q13

i. A stochastic process is weakly stationary if it has constant mean and the covariance is constant for each fixed lag.

ii. The moving average process is $X_n = Z_n + \beta Z_{n-1}$.
The man of the process is $E(X_n) = (1+\beta)\mu. \, this\ is\ constant.$
The variance:
$Var(X_n) = Var(Z_n + \beta Z_{n-1})$
$Var(X_n) = Var(1+\beta^2)\sigma^2$

The covariance
$Cov(X_n, X_{n+1}) = Cov(Z_n + \beta Z_{n-1}, Z_{n+1} + \beta Z_n)$
$Cov(X_n, X_{n+1}) = \beta\sigma^2$

$For\ m > 1, Cov(X_n, X_{n+m}) = \ Cov(Z_n + \beta Z_{n-1}, Z_{n+m} + \beta Z_{n+m-1})$

$= 0$

The covariance at higher lags is 0 since is no overlap between the Z's. The covariance at negative lags is the same as those at the corresponding positive lags. Since none of this expression depends on n, it follows that the process is weakly stationary.

iii. Write the model equation as
$(1 - 1.5B + 5B^2)X = Z$
Where B is s the back-shift operator. The polynomial in B factorizes as
$(1 - b)(1 - 0.5B)$,
Since one of the roots of the polynomial has magnitude 1, the process is NOT stationary.

iv. The process X is ARIMA(1,1,0), so (1−B)X is AR(1). Define the process Y as Y = (1 − B) X, and write this AR(1) process as (1 − .5B)Y = Z .
According to standard formula, the autocorrelation at lag 1 is 0.5.