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Probablity & Stats
Assignment - I

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Q1) $\because x \sim Poi(\theta)$

$\therefore \dfrac{x-\theta}{\sqrt{\theta}} \sim N(0,1)$

also $P[x_1 < \theta < x_2] = 0.95$

$\therefore x \sim Poi(200)$ as $\lambda = 200$

$\therefore 95\%\ C.I = \mu \pm 1.96\sqrt{\sigma}$

$\because \lambda = \mu = \sigma$

$\therefore 95\%\ C.I = 200 \pm 1.96\sqrt{200}$

$= 172.28$ and 227.71

$\therefore$ The interval is $(172.28$ and $227.71)$.

2) Given :

$x_i = 1, 2, 3 \ldots\ldots n$

Sample mean $\bar{x} = \sum_i^n x_i$; Sample variance $s^2 = \dfrac{1}{n-1} \sum_1^n (x_i - \bar{x})^2$

i) $E(\bar{x}) = E\left[\dfrac{\sum x_i}{n}\right] = \dfrac{1}{n}\left[\sum E(x_i)\right] = \dfrac{n\mu}{n} = \mu$

$V(\bar{x}) = V\left[\dfrac{\sum x_i}{n}\right] = \dfrac{1}{n^2} V[E(\bar{x})] = \dfrac{n\sigma^2}{n^2} = \dfrac{\sigma^2}{n}$

ii) $E(s^2) = E\left[\dfrac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x})^2\right]$

$= \dfrac{1}{n-1} E\left[\sum x_i^2 - \bar{x}^2\right] = \dfrac{1}{n-1}\left[E(\sum x_i^2) - E(n\bar{x}^2)\right]$

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$$= \frac{1}{n-1}\left[E(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n}+\mu^2\right)\right]$$

$$= \frac{1}{n-1}\left[n\sigma^2 + \cancel{n\mu^2} - \sigma^2 - \cancel{n\mu^2}\right)$$

$$= \frac{1}{\cancel{n-1}}\left[\sigma^2\cancel{(n-1)}\right]$$

$$= \sigma^2$$

iii) $\because \dfrac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$$\text{var}\left[\frac{(n-1)S^2}{\sigma^2}\right] = \chi^2_{n-1}$$

$$\therefore \frac{(n-1)^2\,\text{var}(S^2)}{\sigma^4} = 2(n-1)$$

$$\therefore \text{var}(S^2) = \frac{2\sigma^4}{n-1}$$

iv) $n = 10$

$\sigma^2 = 100$

$$\therefore \frac{(n-1)S^2}{\sigma^2} = 0.5$$

$$\therefore \frac{9(S^2)}{100} = 0.5$$

$$\therefore S^2 = \frac{0.5 \times 100}{9}$$

$$\therefore S^2 = 5.561$$

3)

No. of claim	0	1	2	3	4
No. of policies	86	75	16	2	1

$X \sim Bin(4, p)$

$\therefore L = P(X=0)^{86} \times P(X=1)^{75} \times P(X=2)^{16} \times P(X=3)^{2} + P(X=4)$

$\therefore L = \left[{}^4C_0\, p^0 (1-p)^4\right]^{86} \left[{}^4C_1\, p^1 (1-p)^3\right]^{75} \left[{}^4C_2\, p^2 (1-p)^2\right]^{16} \left[{}^4C_3\, p^3 (1-p)^1\right]^{2}$
$\left[{}^4C_4\, p^4 (1-p)^0\right]$

$\therefore \ln L = [86(4)[1-p]][75(3)(p)(1-p)][16(4)(p)(1-p)][2(3)p(1-p)][4p] + \text{const}$

$* \therefore \ln L = 344 \ln(1-p) + 75 \ln(p) + 255 \ln(1-p) + 32 \ln p + 32 \ln(1-p)$
$+ 6 \ln p + 2 \ln(1-p) + 4 \ln p + \text{const}$

$\therefore \ln L = 633 \ln(1-p) + 117 \ln p + \text{const}$

$\therefore \frac{d}{dx} \ln L = \frac{117}{p} - \frac{633}{1-p} + 0$

let $\frac{d}{dx}(\ln L) = 0$

$\therefore \frac{117}{p} = \frac{633}{1-p}$

$\therefore p = 0.156$

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ii) $P(X=0) = {}^4C_0\, p^0 (1-p)^4$
$= 91.34$

Similarly
$P(X=1) = 67.53$
$P(X=2) = 18.72$
$P(X=3) = 2.41$
$P(X=4) = 0.12$

$f(x)$	O_i	E_i	$(O_i - E_i)^2/E_i$
$P(X=0)$	86	91.34	0.3123
$P(X=1)$	75	67.53	0.8263
$P(X=2)$	19	18.72	0.3982
			1.5368

$\chi^2_{cal} = 1.5368$
$\chi^2_{tab} = 0.75$

$\because \chi^2_{cal} > \chi^2_{tab}$
Thus we will reject H_0 at 5%.

4) $f(x) = \dfrac{c\beta^2}{(x+\beta)^4} \; ; \; x > 0, \beta > 0$

i) $\int_0^\infty f(x)\, dx = 1$

$c\beta^3 = \int_0^\infty \dfrac{1}{(x+\beta)^4} = 1$

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$$\therefore c\beta^3 = \left[\frac{(x-\beta)^{-3}}{-3}\right]_0^\infty = 1$$

$$\therefore \frac{c\beta^3}{\beta^3} = 1 \quad ; \quad c = 3$$

$$E(X) = \int_0^\infty x \cdot f(x) \cdot dx$$

$$= 3\beta^3 \int \frac{x}{(x+\beta)^4} dx$$

$$= 3\beta^3 \left[\frac{(x+\beta)^{-2}}{-2} - \frac{\beta(x+\beta)^{-3}}{-3}\right]_0^\infty$$

$$= 3\beta^2 \left[\frac{1}{2\beta^2} + \frac{1}{3\beta^2}\right] - \frac{\beta}{2}$$

$$E(X^2) = \int_0^\infty x^2 \cdot f(x)\, dx = 3\beta^3 \int_0^\infty \frac{x^2}{(x+\beta)^4} dx$$

ii) $\beta = X_1, X_2, X_3 \ldots\ldots\ldots X_n$

$E(X) = \bar{X}$ $\qquad \hat{\beta} = 2\bar{X}_n$

$\therefore E(\beta) = E(2\bar{X}_n) = \frac{2n}{n} \times \frac{\beta}{2} = \beta$ i.e unbiased.

iii) $MSE = var(b\bar{X}_n) + [Bias[b\bar{X}_n]]^2$

$$= b^2 \times \frac{3}{4} \times \frac{\beta^2}{n} + [E(b\bar{X}_n) - \beta]^2$$

$$= \frac{\beta^2}{n}\left(\frac{3b^2}{4} + n\left(\frac{b}{2} - 1\right)^2\right)$$

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iv) It is seen that $\hat{B} = 2\bar{x}_n$ is an unbiased estimator
The estimate $b\sqrt{n}$ is (iii) when $b = \dfrac{2}{1+3/n}$ is truly biased
It is also constant.

5) i) $X \sim U(a+b)$

$\therefore \bar{X} = \dfrac{1}{2}(a+b)$

$\therefore S = \dfrac{1}{\sqrt{12}}(b-a)$

$\therefore \sqrt{12} + a = b$

$\therefore \bar{X} = \dfrac{1}{2}[a + \sqrt{12}S + a]$

$\therefore \bar{X} = \dfrac{1}{\cancel{2}}(\cancel{2}a + \cancel{2}\sqrt{3}S)$

$\therefore \hat{a} = \bar{X} - \sqrt{3}S$

ii) $\bar{Y} = \dfrac{1}{2}(a+b)$

$\therefore S = \dfrac{1}{\sqrt{12}}(b-a)$

$\therefore a = b - \sqrt{12}S$

$\therefore \bar{X} = \dfrac{1}{2}[b - \sqrt{12}S + b]$

$\therefore \hat{b} = \bar{X} + \sqrt{3}S$

iii) Sample : 1, 2, 3, 4, 10

$\bar{X} = 20/5 = 4$

$Var = 1/12 \times 81 = 27/4$

$\therefore \hat{a} = \bar{X} - \sqrt{3} \times 27/4 = -7.6913$

$\therefore \hat{b} = \bar{X} + \sqrt{3} \times 27/4 = 15.69134$

Hence the result is false

iv) $f(x) = \frac{1}{b-a} = \frac{1}{b}$

$L = -x \log b$

$\therefore \frac{dL}{db} = \frac{-n}{b} = 0$ as $b \to \infty$

$\frac{d^2 l}{db^2} = \frac{n}{b^2} > 0$, minimum

$\therefore \theta_i = \max . x_i$

6) $H_0 : p = 0.1$
$H_1 : p > 0.1$

i) $P(\text{type } 1) = P[\text{Reject } H_0 \mid H_0 \text{ is } T]$

$= P[X \geq 3 \mid P = 0.1] + P[X = 0 \mid P = 0.1] + P[Y \geq 5 \mid P = 0.1]$
$+ P[X = 1 \mid P = 0.1] \cdot P[Y \geq 4 \mid P = 0.1] + P[Y = 2 \mid P = 0.1]$
$\cdot P[Y \geq 3 \mid P = 0.1]$

$= 0.14727$

$\approx 15\%$

ii) $P(X \geq 3 \mid P = 0.3) + P(X = 0 \mid p = 0.3) = 0.91253 = 91\%$

iii) $P(\text{type II error}) = 1 - 0.9253 = 0.08747$
$\approx 9\%$

iv) $X \sim Bin(120, 1/3)$; $\hat{p} = 40/120 = 1/3$

$\frac{\hat{p} - p}{\sqrt{\frac{p(1-\hat{p})}{n}}} \sim N(0,1)$

$P\left[-1.2810 < \frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} < 1.2816\right]$

$\therefore$ 99% C.I = (0.333 , 0.2782)

v) $\dfrac{X - np_0}{\sqrt{np_0 q_0}} \sim N(0,1)$

vi) P.Q $= \dfrac{(x_1 - x) - (\mu_1 - \mu_2)}{S_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$ $\cancel{S_p^2 = 4027}$

$= 0.2034$

7) i) Public :

$\overline{X_1} = 30$

$S = 8.0195$

$S^2 = 64.3124$

Private :

$\overline{V_1} = 35.06$

$S = 8.2099$

$S^2 = 67.4024$

ii) $H_0 = \overline{X_1} = \overline{X_2}$ $H_1 = \overline{X_1} \neq \overline{X_2}$

$$\frac{(30 - 35.06)}{S_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} = \frac{(30 - 35.06)}{\sqrt{8.1153}\sqrt{2/9}} = -1.3226$$

$$S_p^2 = \frac{8(64.3124) + 8(67.4024)}{16} = 65.8574$$

$Z = (-2.624 , 2.624)$

A_0 not reject H_0

iii) Private hospital don't reasult in higher claim.

9) i) $t_k \sim \frac{Z}{\sqrt{\chi^2_k / k}}$

$Z \sim N(0,1)$

χ^2_k = Chi-square dist

k = degree of freedom

ii) $\mu(t_k) = 0$

$var(t_k) = \frac{k}{k-2}$

iii) $n = 10$

$\mu = 50$

$s^2 = 48.667$

a) $CI = 50 \pm 3.25 \sqrt{\frac{48.667}{10}} = (42.8103, 57.1699)$

b) $CI = 50 \pm 2.5758 \sqrt{\frac{48.667}{10}} = (44.3176, 55.6823)$

10) $n = 1000$

$X \sim Poi(3)$

$N \sim N(3000, 3000)$

i) Type 1 error $= P[H_0 \text{ reject} / H_0 \text{ is true}]$

$= P[X > 3100 / X = 3)$

$= P\left[Z > \frac{3100 - 3000}{\sqrt{3000}}\right]$

$= 1 - P[Z < 1.8257)$

$= 3.89\ \%$

ii) Type 2 error $= P[\text{do not reject } H_0 \mid H_0 \text{ is false}]$

$= P[X < 3100 \mid \lambda = n\lambda]$

$= P\left[X < \frac{3100 - n\lambda}{\sqrt{n\lambda}}\right]$

iii) Power $= P[\text{reject } H_0 \mid H_0 \text{ is false}]$

$= P[X > 3100 \mid \lambda = n\lambda]$

$= 1 - P\left[\frac{3100 - n\lambda}{\sqrt{n\lambda}}\right]$

iv) $n = 2900$

$\therefore$ C.I $= 3 \pm 2.5758 \sqrt{3/2900}$

i.e $(2.9172, 3.0828)$

11) $\Sigma_x = 213,200$

$\Sigma_x^2 = 4919860000$

$\bar{x} = 17767$

S.D $= 10144$

$n = 12$

$\Sigma_x = 213200 - 11000 - 48000 = 154200$

$\Sigma_x' = 154200 + 27500 + 31000 = 213200$

$\bar{x} = 17767$

$\Sigma_x^2 = 4919860000 - 11000^2 - 48000^2 = 2494860000$

$\Sigma_x^{2'} = 24948600 + 27800^2 - 31800^2 = 4243360000$

$S = \frac{1}{\sqrt{11}} \sqrt{4243360000 - 12(17767)^2}$

$\therefore S = 6434.0326$

12) $X_1, X_2, X_3 \ldots\ldots\ldots\ldots\ldots X_n \sim U(0, \theta) \quad a = 0 \; ; \; b = \theta$

i) $E(X) = \theta/2 \quad ; \quad V(X) = \theta^2/12$

$$L = \prod_{i=1}^{n} \left(\frac{1}{\theta}\right) = \frac{1}{\theta^n}$$

$$\ln(L) = \frac{-n}{\theta}$$

$$\frac{d \cdot \ln(L)}{dn} = \frac{-n}{\theta} \quad \text{then} \quad \theta = \text{infinite}$$

Hence CRLB would not be applicable.

ii) $$P[\text{Max } X < Y] = \prod_{i=1}^{n} \left[\int_0^y \frac{dx}{\theta} \right] = \left(\frac{y}{\theta}\right)^n$$

$$E(y^k) = \int y^k \cdot \frac{n}{\theta^n} \cdot y^{n-1} \cdot dy$$

$$= \frac{n}{n+k} \int \frac{n+k}{\theta^n} \cdot y^{n+k-1} \, dy$$

$$= \frac{n\theta^k}{n+k}$$

iv) $$\text{Bias}[\hat{\theta}(c)] = E[\hat{\theta}(c) - \theta]$$

$$= c \cdot E(y) - \theta$$

$$= \frac{cn\theta}{n+1} - \theta$$

$$= \theta\left(\frac{cn}{n+1} - 1\right)$$

iv) $$\frac{d}{dc} MSE[\theta(c)] = \frac{d}{dx}\left[\frac{ln\,\theta^2}{n+2} - \frac{2n\theta^2}{n+1}(1+\theta^2)\right]$$

$$cm = \frac{2n\theta^2}{n+1} \cdot \frac{n+2}{n\theta} \quad \therefore \quad \frac{n+2}{n+1}$$

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$\because \frac{d^2}{dc^2}\left[MSE\left(\hat{\theta}(c)\right)\right] = \frac{2n\theta^2}{n^2} > 0$, minimum value

vi) $\frac{n+2}{n+1} y \rightarrow y$, two estimate became are same $n \rightarrow \infty$.