Assignment I (Unit 1 & 2)

Q1

Ans i.) Arbitrage opportunity is a situation where the investor can make a risk-free profit because of the price mismatch of the same underlying in two different markets.

It means that at time t, the probability of loss is 0 and that of earning profit is positive.

ii.) It states that any 2 portfolios that have exactly the same weights must have the same price. This is the law of one price. If this is not true then the investor can buy at a cheap price & sell at a higher price to earn a riskless profit.

iii.) $K = 120$ (put opt^n) ; $S_0 = 125$, $r = 5\%$

$t = 3/12$; call opt^n = 30

(a) $q = 15\%$ p.a.

By using put-call parity, the value of put opt^n should be :

$$p = C_t + Ke^{-r(T-t)} - S_t e^{-q(T-t)}$$

$$= 30 + 120\, e^{-5\% \times 0.25} - 125\, e^{-15\% \times 0.25}$$

$$p = 28.11$$

(b) If $p = 23$

If the price of the put optn is 23, the investor could buy put optnl at a cheaper price and sell a call optn at a higher price.

For eg:

Sell call - ₹ 30.
Buy put - ₹ 23.

Borrow cash ₹ 100 to buy underlying.

Buy 1 share at ₹ 125

∵ Put-call parity does not hold true,
∴ the investor can make arbitrage profit.

Q2.

Ans) $dX_t = \alpha\mu(T-t)dt + \sigma\sqrt{(T-t)}\,dZ_t$

Let $A_t = \alpha\mu(T-t)$, $B_t = \sigma\sqrt{(T-t)}$ — I

$$dF = \frac{df}{dx} B_t\, dZ_t + \left(\frac{df}{dt} + \frac{df}{dx} A_t + \frac{1}{2}\frac{d^2 f}{dx^2} B_t\right)dt$$

$$= -f B_t\, dZ_t + f \frac{dm}{dt}(T-t)dt - f A_t\, dt + \frac{1}{2} f B_t^2\, dt$$

$$dF = f\left(\frac{dm}{dt}(T-t) - A_t + \frac{1}{2}B_t^2\right)dt - f B_t\, dZ_t$$

For f to be a martingale,

$$\frac{dm}{dt}(T-t) - A_t + \frac{1}{2}B_t^2 = 0.$$

$$\therefore \frac{dm}{dt}(T-t) = A_t + \frac{1}{2}B_t^2$$

$$\therefore \frac{dm}{dt}(T-t) = \alpha\mu - \frac{1}{2}\sigma^2 \quad (\text{from I})$$

Q 3

Ans i) $F(t, x) = e^{-t} x^2$

X is SDE

$$\frac{dX_t}{X_t} = 0.25\,dt + \sigma\, dW_t$$

$dY_t = ?$

$$\frac{df}{dt} = -e^{-t} x^2 = -f$$

$$\frac{df}{dx} = e^{-t}\, 2x$$

By ito's lemma,

$$dY_t = \frac{df}{dt} \cdot dt + \frac{df}{dx} \cdot dX_t + \frac{1}{2} \frac{d^2 f}{dx^2} \cdot \sigma^2 X_t^2 \cdot dt$$

$$= -f \cdot dt + 2e^{-t} X_t\, dX_t + e^{-t} \sigma^2 X_t^2\, dt$$

$$= -Y_t \cdot dt + 2e^{-t} X_t^2 \cdot \frac{dX_t}{X_t} + e^{-t} \sigma^2 X_t^2 \cdot dt$$

$$= -Y_t\, dt + 2Y_t \left[0.25\, dt + \sigma\, dW_t\right] + \sigma^2 Y_t\, dt$$

$$\frac{dY_t}{Y_t} = \left[2 \times \frac{1}{4} - 1 + \sigma^2\right] dt + 2\sigma\, dW_t$$

$$= [\sigma^2 - 0.5]dt + 2\sigma dW_t$$

$$\therefore dY_t = [\sigma^2 - 0.5] Y_t dt + 2\sigma Y_t dW_t$$

(ii) The process is a martingale if drift is 0.
This means $\sigma^2 - 0.5 = 0$, $\sigma^2 = 0.5$

Q.4.

Ans) Let the dividends be denoted by d_1, d_2, d_n

if opt^n is exercised prior to ex-dividend date $= S_{(t)} - k$

if opt^n is not exercised, price drops to $S_{(t)} - d$

The value of the American opt^n is greater than $S_{(t)} - d - Ke^{-r(T-t)}$

It is never optimal to exercise the opt^n if

$$S_{(t)} - d_n - Ke^{-r(T-t)} \geq S_{(t_n)} - k$$

Using this, we have

$$K(1 - e^{-r(T-t)}) = 350(1 - e^{-0.95(0.83 - 0.25)})$$

$$= 18.87$$

and

$$65\left(1-e^{-0.95(0.83-0.25)}\right) = 10.91.$$

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Q 5.

Ans.) Let S be the stock price.

$P(S > 258) = ?$

$S \sim GBM$.

i.e.

$$S_t = S_0 \cdot \exp\left(\mu - \sigma^2/2\right)t + \sigma W_t$$

$$\therefore \ln(S_t) \sim N\left(\left(\ln(S_0) + (\mu - \sigma^2/2)t + \sigma W_t\right), \sigma^2 t\right)$$

$$\therefore \ln(S_t) \sim \phi\left(\ln 254 + (0.16 - 0.35^2/2)0.5, \; 0.35 \times 0.5^{1/2}\right)$$

$$= \phi(5.59, 0.247)$$

$$\therefore P(S > 258) = 1 - N(5.55 - 5.59)/0.247$$

$$= 1 - N(-0.1364)$$

$$= 0.5542.$$

The put optn is exercised if the stock price less than Rs. 258 in 6 months time.

The probability $= 1 - 0.5542$

$= 0.4457.$

Q.6

Ans) $\ln\left(\frac{S_t}{S_0}\right) = ut + \sigma\beta_t$

$$\frac{S_t}{S_0} = e^{ut + \sigma\beta_t}$$

$$y = S_t = S_0 e^{ut + \sigma\beta_t}$$

(i) $\frac{dS_t}{S_t} = x\,d\beta_t + y\,dt$

By ito's lemma,

$$dS_t = \frac{dY}{dt}\cdot dt + \frac{dY}{d\beta_t}\cdot d\beta_t + \cancel{\frac{\sigma^2 S_t}{2}\,dt}\ \frac{d^2Y}{d\beta_t}\cdot\frac{d\beta_t^2}{2}$$

$$dS_t = uS_t\,dt + \sigma S_t\,d\beta_t + \frac{\sigma^2}{2}S_t\,dt$$

$$= S_t\,dt\left(u + \frac{\sigma^2}{2}\right) + \sigma S_t\cdot d\beta_t$$

$$\frac{dS_t}{S_t} = x\,d\beta_t + y\,dt$$

$$x = \sigma \quad , \quad y = u + \frac{1}{2}\sigma^2$$

(ii) $E(S_T) = E\left(S_0\, e^{ut + \sigma B_t}\right)$

$$= S_0 e^{ut} \cdot E\left(e^{\sigma B_t}\right)$$

$B_t \sim N(0,1)$

$E\left(e^{\sigma B_t}\right)$ is the MGF of Normal.

$$= e^{u + \frac{1}{2}\sigma^2 t^2}, \text{ for us, } \sigma^2 u\, t^2 \;\&\; t > \sigma$$

$$= S_0 e^{ut}\left(e^{\frac{1}{2}t\sigma^2}\right)$$

$$= S_0 e^{t\left(u + \frac{1}{2}\sigma^2\right)}$$

$Var(S_T) = E(S_T^2) - [E(S_T)]^2$

$$= E\left(S_0^2\, e^{2ut + 2\sigma B_t}\right) - \left(S_0 e^{t\left(u + \frac{1}{2}\sigma^2\right)}\right)^2$$

$$= S_0^2 \cdot e^{2ut} \cdot E\left[e^{2\sigma B_t} - \left(S_0 e^{t\left(u + \frac{1}{2}\sigma^2\right)}\right)^2\right]$$

$$= S_0^2 e^{2ut}\left(e^{2\sigma^2} - e^{\sigma^2 t}\right)$$

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(iii) $Cov(S_{t1}, S_{t2}) = E(S_{t1}, S_{t2}) - E(S_{t1})E(S_{t2})$

$E(S_{t1}S_{t2}) = S_0 e^{ut_1} E(e^{\sigma B t_1}) \cdot S_0 e^{ut_2} E(e^{\sigma B t_2})$

$S_0^2 e^{u(t_1+t_2)} E(e^{\sigma B t_1 + \sigma[\beta t_1 + (Bt_2 - Bt_1)]})$

The expectation sum is the MGF of Normal Dist^n,

Thus,

$= S_0^2 e^{u(t_1+t_2)} e^{2\sigma^2 t_1} \cdot e^{\frac{1}{2}\sigma^2(t_2-t_1)}$

$= S_0^2 e^{u(t_1+t_2)} \cdot e^{\frac{3}{2}\sigma^2 t_1 + \frac{1}{2}\sigma^2 t_2}$.

Q. 8.

Ans.) (i) Current Stock Price $\rightarrow S_0$

Opt^n price $\rightarrow f$.

Duration $\rightarrow T$

During the term, the stock price will either be $S_0 \cdot u$ or $S_0 \cdot d$, where $u > 1$ & $d < 1$.

Our portfolio : short position in the option and long position in Δ shares.

If upward movement, the value of portfolio becomes $\Delta S_0 u - f_u$.

If downward movement, the value of portfolio becomes $\Delta S_0 d - f_d$.

Two portfolios are equal if,

$$\Delta S_0 u - f_u = \Delta S_0 d - f_d$$

Or $\Delta = \dfrac{f_u - f_d}{S_0 u - S_0 d}$ so that the portfolio is r.f and hence must earn r.f rate of interest.

P.V of portfolio $= (\Delta S_0 u - f_u) e^{-rT}$

The cost of the portfolio is $\Delta S_0 - f$

$\therefore$ portfolio grows at r.f rate,

$$\therefore \Delta S_0 - f = (\Delta S_0 u - f_u)e^{-rT}$$

$$f = \Delta S_0 - (\Delta S_0 u - f_u)e^{-rT}$$

$$\therefore f = e^{-rT}[p f_u + (1-p) f_d]$$

where $p = [e^{rT} - d] / [u - d]$

(ii) The option pricing formula does not involve the probab of stock going up or down, although it is natural to assume that the probability of an upward stock movement increases the value of call optn and dec. in value of put optn decreases when the probab of stock price goes down.

This is because we are calculating the value of optn in terms of the value of underlying stock. It is natural to assume p as probab of up movement and 1-p for down movement. Thus, the above eqn can be intepreted as the value of optn today is the expected future value discounted at the rf rate.

(iii) $E(S_T)$ at time $T = p\, S_0 u + (1-p) S_0 d \cdot 0.5$

or $E(S_T) = p\, S_0 (u-d) + S_0 d - 0.5$

Substituting p from above eqn in part (i),

$E(S_T) = e^{rT} S_0$

i.e stock price grows at a r.f rate

(iv) In a risk-neutral world, individuals do not require compensation for risk. Hence, the exp. return on all securities and optn is the r.f rate. Hence, value of optn is its exp. payoff in a risk-neutral set-up discounted at risk-free rate.

Q9.

Ans) (i) Forward Price = $S \cdot \exp(rt)$

S = stock price
t = delivery time

r = r.f rate

Put-call parity: $c + ke^{-rt} = p + s$

c & p = price of european call & put

k = strike price

S = stock price

t = time to expiry.

Substituting values for simultaneous eqⁿ,

$$13.334 + 70 \cdot \exp(-0.25r) = 0.120 + S$$
$$8.869 + 75 \cdot \exp(-0.25r) = 0.568 + S$$

Solving we get,

$S = 82$, $r = 7\%$

$\therefore$ $F = 82 \cdot \exp(7\% \times 0.25)$

$\therefore$ $F = 83.45$.

(ii.) Let the r.o.i for next k months be r_k

$$\exp(r_6 \times 0.5) = \exp(r_3\ 0.25) \cdot \exp(r_f \times 0.2$$

$$r_3 = 7.1$$

To find r_6, we substitute,

$$2.569 + 90\exp(-0.5 \times r_6) = 7.909 + 82$$

$$\therefore r_6 = 6.1 \ \& \ r_f = 5.1.$$

(iii) Using put-call parity for each row,

$$6.899 + a\exp(-7.1 \times 0.25) = 1.055 + 82.$$

$$b + 80\exp(-7.1 \times 0.25) = 1.789 + 82.$$

$$2.594 + 85\exp(-7.1 \times 0.25) = c + 82.$$

By solving, we get,

$$a = 77.5$$
$$b = 5.177.$$
$$c = 4.119.$$

10.

(i) Rate of interest are assumed zero, the risk - neutral prob is :-

$$q = (1-d) / (u-d)$$

For a recombining tree,

$$d = 1/u$$

$$\therefore q = \left(1 - \frac{1}{u}\right) / \left(u - \frac{1}{u}\right)$$

$$= \frac{1}{u+1}$$

For no arbitrage, we must have $u > 1 > d$

Then, $u > 1 \Rightarrow u + 1 > 2$

$$q = \frac{1}{u+1} < \frac{1}{2}$$

Hence Proved.

(ii) As derived for (i), $q = 1/u+1$.

$$q = \frac{1}{3}$$

$$\therefore u = 2$$

We know that, $u = \exp(\sigma \sqrt{\Delta t})$

Using $u = 2$ and $\Delta t = \frac{1}{12}$,

$\sigma = 240.11\%$

(iii.) Since each step is one month and expiry is 1 yr from now,

$n = 12$.

Further, at time $T = 12$ months the S.P will be $S_0 u^k d^{n-k}$ with risk-neutral probab

${}^nC_k q^k (1-q)^{n-k}$, where q is the upstep probab and u is the upstep size

$q = 1/3$.
$u = 2$.
$d = 1/2$.

We know that the derivative has a payoff $\sqrt{\frac{S_T}{S_0}}$

at time $T = 12$ months.

The C.P is $P = \sum_{k=0}^{n} \sqrt{\frac{S_T}{S_0}} \cdot \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k}$

$$= \sum_{k=0}^{n} \sqrt{\frac{S_0 u^k d^{n-k}}{S_0}} \cdot \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k}$$

$$= \sum_{k=0}^{n} \sqrt{u^k \cdot d^{n-k}} \; \frac{n!}{k!(n-k)!} \cdot q^k (1-q)^{n-k}$$

$$= \sum_{k=0}^{n} u^{\frac{k}{2}} \cdot {}^{\frac{n-k}{2}} \cdot \frac{n!}{k!(n-k)!} \, q^{k} (1-q)^{n-k}$$

$$= \sum_{k=0}^{n} 2^{k/2} \cdot \left(\frac{1}{2}\right)^{\frac{n-k}{2}} \cdot \frac{n!}{k!(n-k)!} \left(\frac{1}{3}\right)^{k} \left(\frac{2}{3}\right)^{n-k}$$

$$= \sum_{k=0}^{n} 2^{k-\frac{n}{2}} \cdot \frac{n!}{k!(n-k)!} \cdot \frac{2^{n-k}}{3^{n}}$$

$$= \frac{2^{\frac{n}{2}}}{3^{n}} \cdot \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}$$

$$= \frac{2^{\frac{n}{2}} \cdot 2^{n}}{3^{n}}$$

$$= \left(\frac{2 \cdot \sqrt{2}}{3}\right)^{n}$$

$$= \left(\frac{2\sqrt{2}}{3}\right)^{12}$$

$$= 0.49327$$

Q. 11.

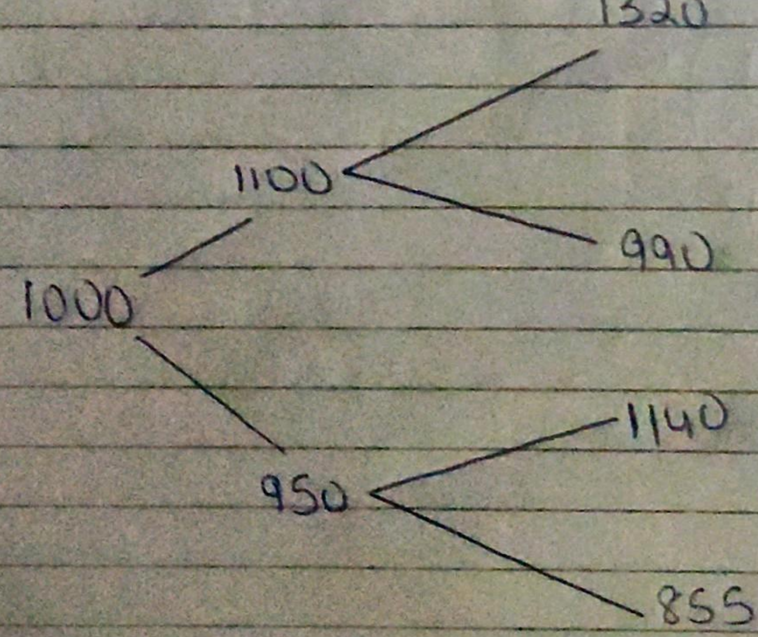
Ans (i) In a recombining binomial tree is one in which the sizes of up and down steps is the same under all states and across all time intervals

It therefore follows that the risk neutral probab 'q' is also constant at all times & in all states.

The main adv is that it has only (n+1) possible states of time as opposed to 2^n possible states in a non-recombining t

The main disadv is that it implicitely assum the the volatility & drift parameters of th underlying are constant over time

(b.)(i)



$$q_1 = \frac{e^{(0.0175)} - 0.95}{1.10 - 0.95}$$

$$= 0.4510.$$

$$q_2 = \frac{e^{0.025} - 0.90}{1.20 - 0.90}$$

$$= 0.41772.$$

Put payoff at 4 states are : 0, 0, 0 & 95.

Working backwards, the value of option $V_1(1)$ following an up step over the first 3 months is:

$$V_1 \cdot \exp^{0.025} = [0.41772 \times 0] + [0.58228 \times 0]$$

$$\therefore V_1(1) = 0.$$

The value of option following a down step over the first 3 months is:

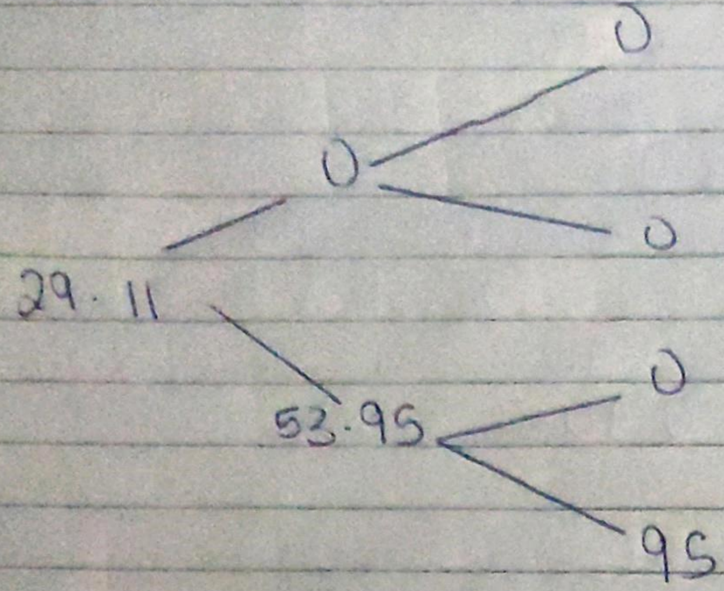
$$V_2(2) \cdot \exp^{0.025} = [0.41772 \times 0] + [0.58228 \times 95]$$

$$= 53.9508.$$

The Current Value is:

$$V_0 \exp(0.0175) = [0.4510 \times 0] + [0.5490 \times 53.9508]$$

$$= 29.105.$$



(ii) While the proposed modification would produce more accurate valuation, a lot more parameters would have to be specified.

The new tree would be $2^6 = 64$ nodes. This would be computatitionally cumbersome

An alternate model that might be more efficient would be a 6-step tree which would have only 7 nodes in the expiry column.

Ans.) (i) $Z(t)$ is a standard Brownian,

(a.) $dU(t) = 2dZ_t - 0$

$= 0dt + 2dZ_t$

Thus, $U(t)$ has zero drift.

(b.) $dV(t) = d[Z(t)]^2 - dt$

$d[Z(t)]^2 = 2Z(t)\, dZ_t + dt$

Thus, $dV(t) = 2Z_t \cdot dZ_t$. The stochastic process $V(t)$ has 0 drift.

(c.) $dW(t) = d[t^2 Z(t)] - 2t\, Z(t)dt$.

Because $d[t^2 Z(t)] = t^2 dZ(t) + 2t\, Z(t)dt$

$dW(t) = t^2\, dZ(t)$

Thus, the process $W(t)$ has 0 drift.

Q 13

Ans) Let $\frac{S_t}{S_0} \sim LN\left[\left(u - \frac{1}{2}\sigma^2\right)t, \sigma^2 t\right]$

expected return = u.
volatality = σ.

Expected value at first time step on tree = $qS_0u + (1-q)S_0d$

$$S_0 e^{u\delta t} = qS_0u + (1-q)S_0d$$

$$q = (e^{u\delta t} - d) / (u-d)$$

S.D of stock price = $\sigma\sqrt{\delta t}$

The variance is :

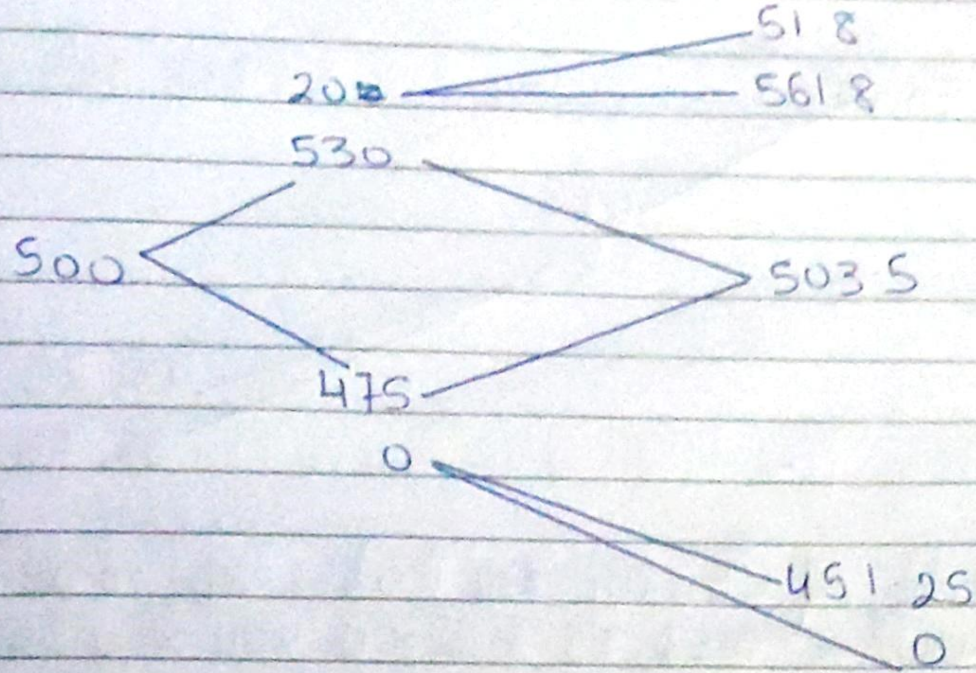
$$qu^2 + (1-q)d^2 - [qu + (1-q)d]^2 = \sigma^2 \delta t$$

$$e^{u\delta t}(u+d) - ud - e^{2u\delta t} = \sigma^2 \delta t$$

$$u = e^{\sigma\sqrt{\delta t}} \text{ and } d = e^{-\sigma\sqrt{\delta t}}$$

Q.14.

(i) Payoff Diagram for European call



$u = 1.06$

$d = 0.95$

$r = 0.05$

$t = 0.25$

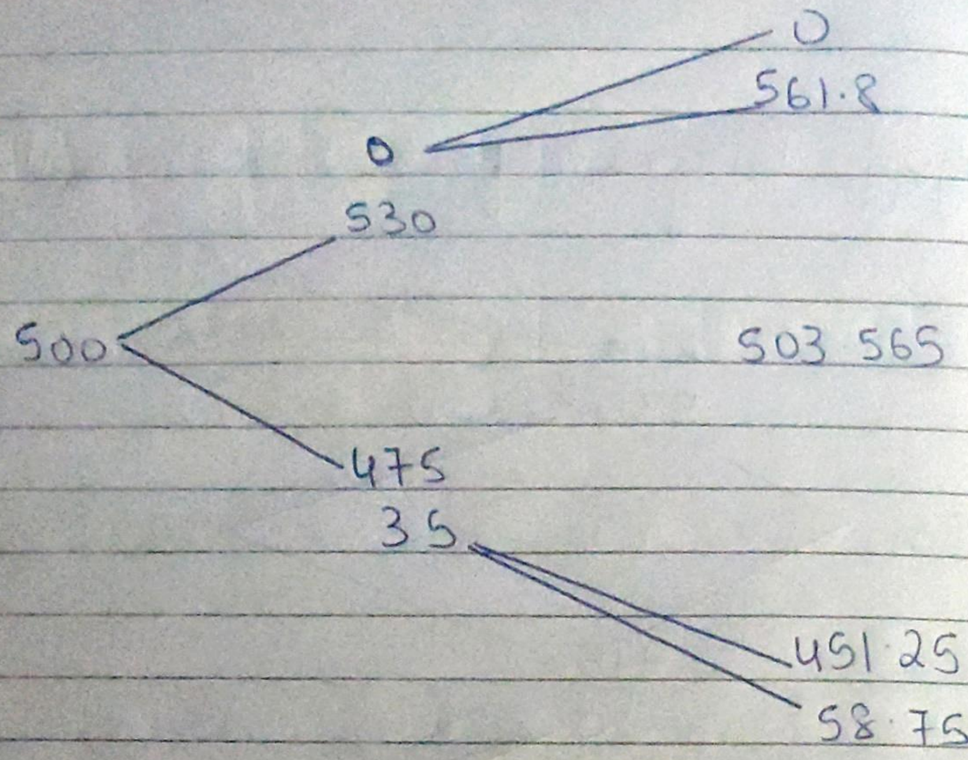
$p = \exp(rT) - d \mid (u-d) = 0.5689$

$1-p = 0.4311$

Value of 6 month Europen call $= \exp(-0.5 \times 0.05) \times 51.8 \times 0.5689^2$

$= 16.351$

(ii) Payoff for Put :



Value of European Put = exp(−0.05 × 0.5) ×
[0.4311^2 × ~~0~~58.75 + 2 × 0.4311 × 0.5689 × 6.50]

= 13.759

Using Put-call parity,

Value of put + stock = 513.759
Value of call + discounted price = 513.759

(iii.) For American Optⁿ, the value at final node is same for a European Opt^n.

Value at Node B = 6.50 * exp(−0.25 × 0.05) × 0.4311 = 2.767

C = exp(−0.25 × 0.05)(6.50 × 0.5689 + 58.75 × 0.4311)

Value of American put option at $t = 0$

$$\exp(-0.05 \times 0.25)(35 \times 0.4311 + 2.767 \times 0.5869)$$
$$= 16.46$$

(iv) real rate of return = 9%

$$p = 0.6614$$

Exp payoff $= 20 \times 0.6614$
$= 13.228$.

Unfortunately, it is not easy to know the correct discount rate to apply in real-world. Risk-neutral valuation solves this problem

Q. 15

Ans) (i) Let $f = f(S_t, t) = S^k$

$\frac{df}{dS} = kS^{k-1}$, $\frac{d^2f}{dS^2} = k(k-1)S^{k-2}$

$\frac{df}{dt} = 0$.

Using ito calculus,

$$df = kS^{k-1}dS_t + \frac{1}{2}k(k-1)S^{k-2}(dS_t)^2$$

$$= f\left[k\mu + \frac{1}{2}k(k-1)\sigma^2\right]dt + f\left[k\sigma\right]dW_t$$

Hence $f = S^k \sim GBM$

with drift $= k\mu + \frac{1}{2}k(k-1)\sigma^2$

volatility $= k\sigma$.

$f_t = f_0 e^{(\mu' - \frac{1}{2}\sigma'^2)t + \sigma' W_t}$

This means $\frac{f}{f_0} \sim LN\left[\left(\mu' , \frac{1}{2}\sigma'^2\right)t, \sigma'^2 t\right]$

$E(f) = f_0 e^{\mu' t}$, $V(f) = f_0 e^{2\mu' t}\left(e^{\sigma'^2 t} - 1\right)$

(ii) Let $f = f(t, S_t) = e^{-rt} S_t$

$\frac{\partial f}{\partial t} = -re^{-rt} S_t$, $\frac{\partial f}{\partial S} = e^{-rt}$, $\frac{\partial^2 f}{\partial S^2} = 0$

By Ito calculus,

$$df = -re^{-rt} S_t \, dt + e^{-rt} dS_t$$

$$= (u - r) f \, dt + \sigma f \, dW_t$$

which is a martingale if and only if $u = r$.

(iii) S^k to be a martingale $= ku + \frac{1}{2} k(k-1)\sigma^2 = r$