Probability & Statistics

(1) Let X_i be claim size on home insurance
Let Y_i be claim size on car insurance

So,

$$X \sim N(800, 10000)$$
$$Y \sim N(1200, 90000)$$

$$P(y_1 + y_2 + y_3 > x_1 + x_2 + x_3 + x_4 + 800)$$

$$P(Y_1 + Y_2 + Y_3 - (X_1 + X_2 + X_3 + X_4) > 800)$$

$$(Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) \sim N(3 \times 1200 - 4 \times 800, \; 300^2(3) + 100^2(4))$$

$$\sim N(400, 310000)$$

$$P\left(Z > \frac{800 - 400}{\sqrt{310000}}\right)$$

$$P(Z > 0.71842) = 1 - P(Z < 0.71842)$$
$$1 - P(Z < 0.72)$$
$$= 1 - 0.76424$$
$$= 0.23576$$

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(2) $N \sim$ Negative Binomial (K, P)

$P(N, n) = {}^{n+2}C_{n} (0.9)^3 (0.1)^n$

$= {}^{(n+3)-1}C_{n-1} (0.9)^3 (0.1)^n$

Here, $k = 3$ $p = 0.9$

$M_y t = M_N'(\log M_x t)$

$X \sim$ Gamma $(6, 2)$

$Y =$ Total claim amounts

i) MGF of y $\quad M_y(t) = E[E(e^{yt}/N=n]$

$E(e^{yt}/N=n) = E(e^{x_1 t + x_2 t + x_3 t \dots x_n t})$

$= E(e^{tx_1}) + E(e^{tx_2}) + E(e^{tx_3}) \dots, E(e^{tx_n})$

$= \left(1 - \frac{t}{2}\right)^{-6n}$

So, $E(E(e^{yt}/N=n)) = E\left(\left(1 - \frac{t}{2}\right)^{-6n}\right)$

$= E\left(e^{N \log\left(1-\frac{t}{2}\right)^{-6}}\right)$

$= M_N\left(\log\left(1-\frac{t}{2}\right)^{-6}\right)$

$M_N(t) = \left(\frac{pe^t}{1-qe^t}\right)^k = \left(\frac{pe^{\log\left(1-\frac{t}{2}\right)^{-6}}}{1-qe^{\log\left(1-\frac{t}{2}\right)^{-6}}}\right)^K$

$$\left[\frac{p\left(1-\frac{t}{2}\right)^{-6}}{1-q\left(1-\frac{t}{2}\right)^{-6}}\right]^3 = \left[\frac{(0.9)\left(1-\frac{t}{2}\right)^{-6}}{1-0.1\left(1-\frac{t}{2}\right)^{-6}}\right]^3$$

$$V(Y) = E(N)V(X) + (E(X))^2 \cdot V(N)$$

$$= \left[\frac{3}{0.9} \times \frac{6}{4}\right] + \left(3\right)^2 \times \frac{3(0.1)}{(0.9)^2}$$

$$= 2.887$$

(3) $f(x) = \lambda e^{-\lambda(x-\alpha)}$

i) $MGF = E(e^{tx}) = \int_{\alpha}^{\infty} e^{tx} \lambda e^{-\lambda(x-\alpha)} \, dx$

$$M_X t = \lambda \int_{\alpha}^{\infty} e^{tx} e^{-\lambda x} e^{\lambda\alpha} \, dx \Rightarrow e^{\lambda\alpha} \lambda \int_{\alpha}^{\infty} e^{-(\lambda-t)x} \, dx$$

$$\frac{e^{\alpha\lambda}\lambda}{t-\lambda}\left[e^{-(\lambda-t)x}\right]_{\alpha}^{\infty} \Rightarrow \frac{e^{\lambda\alpha}\lambda}{t-\lambda}\left[-e^{-(\lambda-t)\alpha}\right]$$

$$= \frac{e^{\lambda\alpha}\lambda}{\lambda-t} e^{-\lambda\alpha} e^{t\alpha} = e^{t\alpha}\frac{\lambda}{\lambda-t}$$

ii) $M_X(t) = \frac{\lambda e^{t\alpha}}{\lambda - t} \Rightarrow \lambda e^{t\alpha}(\lambda-t)^{-1}$

$$M_X'(t) = \lambda\left[e^{t\alpha}(\lambda-t)^{-2}(1) + (\lambda-t)^{-1} e^{t\alpha}(\alpha)\right]$$

$$= \lambda\left[(\lambda-t)^{-1} e^{t\alpha}(\alpha) + e^{t\alpha}(\lambda-t)^{-2}\right]$$
$$= \lambda e^{t\alpha}(\lambda-t)^{-2}\left[\alpha(\lambda-t)+1\right]$$

$$= \frac{1}{\lambda}\left[\lambda\alpha+1\right] = \frac{\lambda\alpha+1}{\lambda}$$

$$M_x''(t) \Rightarrow \lambda\left[\alpha e^{t\alpha}(\lambda-t)^{-2} + \alpha^2(\lambda-t)^{-1} e^{t\alpha} + e^{t\alpha}(2)(\lambda-t)^{-3} + (\lambda-t)^{-2} e^{t\alpha}\right]$$

Putting $t=0$

$$\lambda\left[\alpha(\lambda)^{-2} + \alpha^2(\lambda)^{-1} + (2)(\lambda)^{-3} + \alpha(\lambda)^{-2}\right]$$

$$\lambda\left[\frac{2\alpha}{\lambda^2} + \frac{\alpha^2}{\lambda} + \frac{2}{\lambda^3}\right]$$

$$E(x^2) = \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2}$$

$$V(x) = E(x^2) - \left[E(x)\right]^2 = \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left[\frac{\alpha\lambda+1}{\lambda}\right]^2$$

$$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left(\alpha^2 + \frac{1}{\lambda^2} + \frac{2\alpha}{\lambda}\right)$$

$$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \alpha^2 - \frac{1}{\lambda^2} - \frac{2\alpha}{\lambda} = \boxed{\frac{1}{\lambda^2}}$$

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(4) $G_Y(t) = \left(\frac{p}{1-qt}\right)^k$

$G_Y(t) = \sum t^y \times P(Y=y)$

$P(Y=4) = \frac{\sqrt{k+4}}{\sqrt{5}\sqrt{k}} p^k q^k$

$= \frac{\sqrt{k+4}}{4!\sqrt{k}} p^k q^k = \frac{(k+3)!}{(k-1)!\,4!} p^k (1-p)^k$

(5)
(6) $X \sim \text{Poisson}(\lambda)$ $Y \sim \text{Poisson}(\mu)$

$M_X(t) = e^{\lambda(e^t-1)}$ $M_Y(t) = e^{\mu(e^t-1)}$

So, $Z = X - Y$

$M_Z(t) = E(e^{tZ}) = E(e^{t(X-Y)}) = E(e^{tX} \cdot e^{-tY})$

$= E(e^{tX}) \times E(e^{-tY}) = \frac{E(e^{tX})}{E(e^{tY})}$

$= \frac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}} = e^{(\lambda-\mu)(e^t-1)}$

MGF of $Z = e^{(\lambda-\mu)(e^t-1)} \sim \text{Poisson}(\lambda-\mu)$

Let $Z = X+Y$

$E(e^{tZ}) = e^{\lambda(e^t-1)} \times e^{\mu(e^t-1)}$

$= e^{(\lambda+\mu)(e^t-1)}$

So, by uniqueness $Z \sim \text{Poisson}(\lambda+\mu)$

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		X			
		1	2	3	4
Y	1	0.2	0	0.05	0.15
	2	0	0.3	0.1	0.2

i) $V(X|Y=2) = E(X^2|Y=2) - [E(X|Y=2)]^2$

$= E(X^2|Y=2] = \sum_x \frac{x^2 P(X=x, Y=2)}{P(Y=2)}$

$x=1$

$\frac{(1)^2 P(X=1, Y=2)}{P(Y=2)} + \frac{(2)^2 P(X=2, Y=2)}{P(Y=2)} + \frac{(3)^2 P(X=3, Y=2)}{P(Y=2)}$

$+ \frac{(4)^2 P(X=4, Y=4)}{P(Y=2)}$

$= \frac{(1)(0)}{0.6} + \frac{4\times 0.3}{0.6} + \frac{9\times 0.1}{0.6} + \frac{16\times 0.2}{0.6}$

$= 8.834$

$E(X|Y=2) = \sum_x \frac{x P(X=x, Y=2)}{P(Y=2}$

$\frac{(1)(0)}{0.6} + \frac{2(0.3)}{0.6} + \frac{(3)(0.1)}{0.6} + \frac{4(0.2)}{0.6}$

$= \frac{0.6+0.3+0.8}{0.6} = 2.834$

$= 0.807$

ii) $f(u,v) = \frac{48}{67}(2uv - u^2) \quad 0<u<1 ; \frac{v}{2}<v<2$

$E(U/V = v) =$

$\frac{f(u,v)}{f(v)} =$

$f(v) = \int_0^1 \frac{48}{67}(2uv - v^2)\,du = \frac{48 \times 2}{67}\left[\frac{u^2 v}{} - \frac{u^3}{3}\right]_0^1$

$\frac{96}{67}\left[v - \frac{1}{3}\right] = \frac{32}{67}\left[3v - 1\right] = f(v)$ //

$\frac{f(u,v)}{f(v)} = \frac{32}{67} \frac{(2uv^2 - v^2]}{\cancel{32}(3v-1)} \times 67$

$= \frac{1}{3v-1}(2uv^2 - u^2) \quad 2u^2v^2 - u^3$

$= E(U/V=v) = \frac{1}{3v-1}\int_0^1 u\,(2uv^2 - u^2)\,du$

$\frac{1}{3v-1}\left[\frac{2u^3v^2}{3} - \frac{u^4}{4}\right] = \frac{1}{3v-1}\left[\frac{2v^2}{3} - \frac{1}{4}\right]$

$= \frac{1}{3v-1}\left[\frac{8v^2-3}{12}\right] = \frac{8v^2-3}{12(3v-1)}$

9) $E(X) = 3$, $V(X) = 4$

$E(Y) = 4$, $V(Y) = 1$

$Cov(X,Y) = -0.3\sqrt{4} = -0.6$

$Z = X + Y$

i) $Cov(X, X+Y) = Cov(X,X) + Cov(X,Y)$

$= V(X) + Cov(X,Y)$

$= 4 + (-0.6) = 3.4$

ii) $Var(Z) = Cov(X+Y, X+Y)$

$= Cov(X,X) + 2Cov(X,X) + Cov(Y,Y)$

$= V(X) + 2Cov(X,Y) + V(Y)$

$= 4 + 2(-0.6) + 1$

$= 5 - 1.2 = 3.8$

Q10) $f(x,y) = K x^{-\alpha} e^{-y/\beta}$

$1<x<\infty, 1<y<$

$\alpha>1, \beta>0$

$= K \int_1^\infty \int_1^\infty x^{-\alpha} e^{-y/\beta}\, dx\, dy = 1$

$K \int e^{-y/\beta} \left[\frac{x^{-(\alpha-1)}}{-\alpha+1} \right]_1^\infty = 1$

$\frac{K}{1-\alpha} \int e^{-y/\beta} [-1] = \frac{K}{\alpha-1} \int e^{-y/\beta} dy = 1$

$\frac{K}{\alpha-1} \left[\frac{e^{-y/\beta}}{(-1/\beta)} \right]_1^\infty = \frac{-K}{\alpha-1} (\beta)(e^{+1/\beta}) = 1$

$K = \frac{\alpha-1}{(\beta)(e^{-1/\beta})}$

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