SRM-2 (Assignment 2)

1. Question 1

i. Writing the state space in the order {Bid (B), Offer(O)} the generator matrix is:

$$\begin{array}{c} \\ B \\ O \end{array}\begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

ii. The holding times are exponentially distributed with the parameter λ in state B, and μ in state O.

iii.
$$\frac{d\,{}_tP_s^{BB}}{dt} = -\lambda \cdot {}_tP_s^{BB} + \mu \cdot {}_tP_s^{BO}$$

$$\frac{d\,{}_tP_s^{BO}}{dt} = \lambda \cdot {}_tP_s^{BB} - \mu \cdot {}_tP_s^{BO}$$

iv. We have a two state Model so:

$${}_tP_s^{BB} + {}_tP_s^{BO} = 1$$

Substituting.

$$\frac{d\,{}_tP_s^{BB}}{dt} = -\lambda \cdot {}_tP_s^{BB} + \mu\left(1 - {}_tP_s^{BB}\right)$$

$$d\left[\exp((\lambda+\mu)t) \cdot {}_tP_s^{BB}\right] = \mu \cancel{(1-{}_tP_s^{BB})} \exp((\lambda+\mu)t)$$

hence.

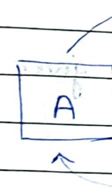
$$\exp((\lambda+\mu)t) \cdot {}_tP_s^{BB} = \frac{\mu}{\lambda+\mu}\exp((\lambda+\mu)t) + \text{constant}$$

Since the process is in State B'd at time s (t=0), the constant is $\frac{\lambda}{\mu+\lambda}$

and thus ${}_{t}P_{s}^{BB} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \cdot \exp(-(\lambda+\mu)t)$

2. Question 2

i.



ii. $\frac{d}{dt} P_{AA}(t)$

$\frac{d}{dt} P_{AB}(t$

$\frac{d}{dt} P_{AC}($

iii. To stay

$\frac{d}{dt} P_{\overline{AA}}($

$P_{\overline{AA}}(t$

So for

We can

$P(Poi($

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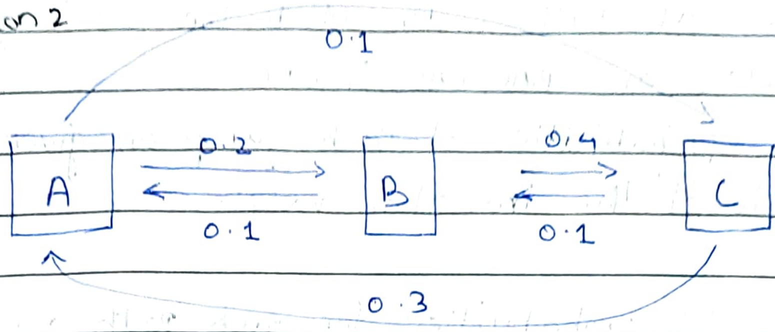
2. Question 2

i.



ii. $\frac{d}{dt} P_{AA}(t) = -0.3\, P_{AA}(t) + 0.1\, P_{AB}(t) + 0.3\, P_{AC}(t)$

$\frac{d}{dt} P_{AB}(t) = 0.2\, P_{AA}(t) - 0.5\, P_{AB}(t) + 0.1\, P_{AC}(t)$

$\frac{d}{dt} P_{AC}(t) = 0.1\, P_{AA}(t) + 0.4\, P_{AB}(t) - 0.4\, P_{AC}(t)$

iii. To stay in state A the equation reduces to:

$\frac{d}{dt} P_{\overline{AA}}(t) = -0.3\, P_{\overline{AA}}(t)$, which has the solution

$P_{\overline{AA}}(t) = \exp(-0.3t)$

So for $t = 2$, we have $\exp(-0.6) = 0.5488$.

OR

We can model this as Poisson with ~~equation~~ parameter $(0.1 + 0.2) \times 2 = 0.6$

$P(Poi(0.6)) = \frac{e^{-0.6}\, 0.6^0}{0!}$

$= 0.5488$

iv. The only paths under which the third jump is into State C are BAC, CAC & CBC.

The propabilities of each jump are given by the ratio of the transition rates.

So, the probabilities for each path are:

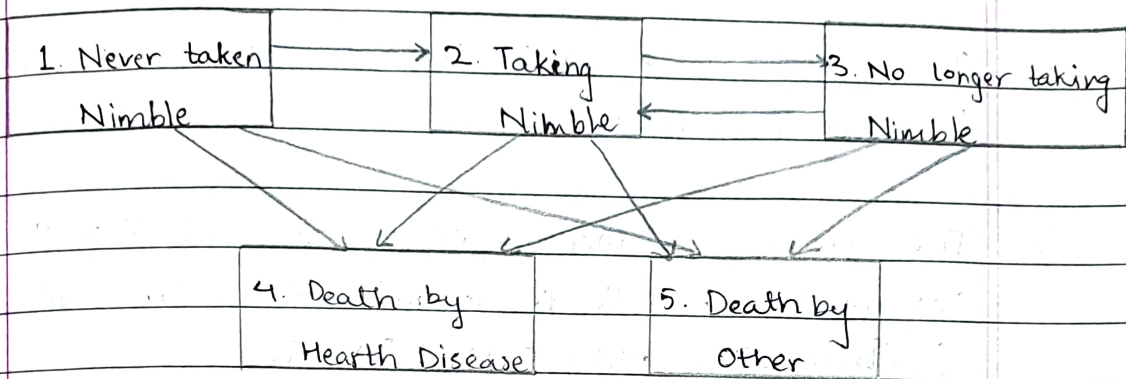
$$BAC = \frac{2}{3} \times \frac{1}{5} \times \frac{1}{3} = \frac{2}{45}.$$

$$CAC = \frac{1}{3} \times \frac{\cancel{3}}{4} \times \frac{1}{\cancel{3}} = \frac{1}{12}$$

$$CBC = \frac{1}{3} \times \frac{1}{\cancel{4}} \times \frac{\cancel{4}}{5} = \frac{1}{15}$$

Sum $= 7/36 = 0.194$

5. Question 3

i. 

ii. Using the Markov assumptions or Champan Kolmogorov equiation

$${}_{dt+t}P_x^{34} = {}_tP_x^{31}\,{}_{dt}P_{x+t}^{14} + {}_tP_x^{32}\,{}_{dt}P_{x+t}^{24} + {}_tP_x^{33}\,{}_{dt}P_{x+t}^{34} + {}_tP_x^{34}\,{}_{dt}P_{x+t}^{44} + {}_tP_x^{35}\,{}_{dt}P_{x+t}^{54}$$

Since ${}_{dt}P_{x+t}^{54} = {}_tP_x^{31} = 0$.

$${}_{dt+t}P_x^{34} = {}_tP_x^{32}\,{}_{dt}P_{x+t}^{24} + {}_tP_x^{33}\,{}_{dt}P_{x+t}^{34} + {}_tP_x^{34}\,{}_{dt}P_{x+t}^{44}$$

Given that ${}_{dt}P_{x+t}^{44} = 1$.

& Assuming that for dt

$${}_{dt}P_{x+t}^{ij} = \mu_{x+t}^{ij}\,dt + o(dt) \qquad i \neq j$$

where $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$, substituting.

$${}_{dt}P_x^{34} = {}_tP_x^{32}\,\mu_{x+t}^{24}\,dt + {}_tP_x^{33}\,\mu_{x+t}^{34}\,dt + {}_tP_x^{34} + o(dt)$$

So that, $${}_{dt+t}P_x^{34} - {}_tP_x^{34} = {}_tP_x^{32}\,\mu_{x+t}^{24}\,dt + {}_tP_x^{33}\,\mu_{x+t}^{34}\,dt + o(dt)$$

4. Question 4

i. The mean is equal to the parameter, so there are 3 calls per hour.

ii. The process is memoryless so the fact that Fred has not had a call for 15 minute is irrelevant.
E(time until next call) = 20 min

iii. The probability of zero calls in time 0.5 hours.
Using $P_j(t) = e^{-\lambda t} (\lambda t)^t / j!$

OR

$$P_0(0.5) = \frac{e^{-1.5}(1.5)^0}{0!}$$

$= 0.2231.$

iv. The expected time that Fred is on the phone is the expected number of calls times the ~~exp~~. length of a call.

Per hour this is 3 calls × 7 minutes = 21 minutes

So, the probability that the phone is engaged is $\frac{21}{60} = 0.35$

· Question 5

Q5.

i. Using Markov Assumption or Chapman Kolmogrov equation

$$P_{HH}(x, t+dt) = P_{HH}(x,t)\, P_{HH}(t, t+dt) + P_{HS}(x,t) \cdot P_{SH}(t, t+dt) + P_{HD}(x,t) \cdot P_{DH}(t, t+dt)$$

But $P_{DH}(t, t+dt) = 0$;

so,

$$P_{HH}(x, t+dt) = P_{HH}(x,t) \cdot P_{HH}(t, dt+t) + P_{HS}(x,t)\, P_{SH}(t, t+dt)$$

Assuming that, for small dt

$\cancel{P_{ij}(x, t+dt) = P_{HH}(x,t)\, P_{HH}(t, t+dt) + PHS(x,t)\, P_{Si}}$

$$P_{ij}(t, t+dt) = \lambda_{ij}(t)\,dt + o(dt)$$

$$P_{ij}(t, t+dt) = 1 + \lambda_{ii}(t)\,dt + o(dt).$$

OR

$$P_{ij}(t, t+dt) = 1 - \sum_{j \neq i} \lambda_{ij}(t)\,dt + o(dt)$$

where the λ_s are the instantenous transition rates and $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$, s

then substituting

$$P_{HH}(x, t+dt) = P_{HH}(x,t)(1-\sigma(t)dt - \mu(t)dt) + P_{HS}(x,t)p(t, c_t) + o(dt).$$ Q6

so that

$$P_{HH}(x, t+dt) - P_{HH}(x,t) = P_{HH}(x,t)(-\sigma(t) - \mu(t))dt + P_{HS}(x,t)\, p(t, c_t)dt + o(dt)$$

and hence.

$$\frac{d}{dt} P_{HH}(x,t) = \lim_{dt \to 0} \frac{P_{HH}(x, t+dt) - P_{HH}(x,t)}{dt}.$$

$$= P_{HH}(x,t)(-\sigma(t) - \mu(t)) + P_{HS}(x,t)\, p(t, c_t)$$

ii. The equation simplifies when considering $P_{\overline{HH}}(t)$ to

$$\frac{d}{dt} P_{\overline{HH}}(0,t) = -(\sigma(t) + \mu(t)) P_{\overline{HH}}(t)$$

$$\frac{1}{P_{\overline{HH}}(0,t)} \frac{d}{dt} P_{\overline{HH}}(0,t) = -(\sigma(t) + \mu(t))$$

$$= \frac{d}{dt} \ln P_{\overline{HH}}(t).$$

Integrate b.t.s.

$$\left[\ln P_{\overline{HH}}(0,t)\right]_0^t = \int_0^t -(\sigma(s) + \mu(s))ds, \text{ as } P_{\overline{HH}}(0) = 1.$$

$$P_{\overline{HH}}(0,t) = \exp - \left(\int_{s=0}^{t} (\sigma(s) + \mu(s))\, ds\right)$$

Q6 All three processes have as discrete state space. A Markov Chain and Markov Jump Chain both operate in discrete time but a Markov Jump Process operates in continuous time. All ~~th~~ have the Markov Property, which is that the future development of the process can be predicted from its present state alone, without reference to its past history.

$$P[X_t \in A \mid X_{s_1} = x_1,\ X_{s_2} = x_2, \ldots, X_{s_n} = x_n, X_s = x] =$$

$$P[X_t \in A \mid X_s = x],$$

for all times $s_1 < s_2 < \ldots < s_n < s < t$, all states $x_1, x_2, x_3 \ldots x_n$, x in S and all subsets A of S.

If a Markov Jump Process X is examined only at the times of its transistions, the resulting process is called the Jump Chain associated with X.

The Jump Chain obeys the Markov Property and behaves as a Markov chain except when the Jump Chain encounters an absorbing state. From that time, it makes no further transitions, implying that time stops for the Jump Chain.
The Jump Chain associated with X takes the same path through the state space as X does. However, questions about the times taken to visit a state are likely to have a different answers for X and for the Jump Chain associated with X. The Markov Jump Chain & the Markov Chain are xpressed in terms of probabilities whereas the Markov Jump Process is expressed in

terms of rates.

The Markov Chain can have loops in each state, the Markov Jump process cannot and the Markov Jump Chain only has loops absorbing states

Q7 Question 7:

i. The maximum likelihood estimates of the transition intensity from state i to state j is the number of transitions from state i to state j divided by the total waiting time in state i. To estimate the transition intensities exactly we therefore need the total time spent in each state

OR

the entry and exit times for each individual for each state, and the total number of transitions of each type made.

ii. $$\frac{d}{dt} P_{AA}(s,t) = -P_{AA}(s,t)\mu$$

$$\frac{d}{dt} P_{AT}(s,t) = P_{AA}(s,t)\mu$$

OR

$$\frac{d}{dt} p(s,t) = p(s,t) M, \quad M = \begin{bmatrix} -\mu & \mu \\ 0 & 0 \end{bmatrix}, \text{ Active, theft.}$$

OR

$$P_{AA}(s,t) = \exp\left(-\int_{u=s}^{t} \mu \, du\right)$$

$$P_{AT}(s,t) = \int_{u=0}^{t} P_{AA}(s,u) \cdot \mu \cdot du$$

iii. Measure from time zero, i.e., $s=0$ and drops s from notation.

$$\frac{1}{P_{AA}(t)} \frac{d}{dt} P_{AA}(t) = -\mu$$

$$\frac{d}{dt}\left(\ln(P_{AA}(t))\right) = -\mu$$

hence $P_{AA}(t) = \exp(-\mu t + c)$.

As $P_{AA}(0) = 1$, $c = 0$, so

$$P_{AA}(t) = \exp(-\mu t)$$

A claim occurs with cost £C if moves to state "Theft Claim". Hence the expected cost is $C(1\ \exp(-\mu T))$

iv.

$$\begin{tikzcd} \boxed{\text{Active Policy}} \arrow[r, "\mu"] \arrow[d, "\lambda"] & \boxed{\text{Theft Claim}} \\ \boxed{\text{Lapsed}} & \end{tikzcd}$$

v. $$\frac{d}{dt} P_{AA}(t) = -P_{AA}(t)(\mu+\lambda), \quad P_{AA}(t) = \exp(-(\mu+\lambda)t)$$

$$\frac{d}{dt} P_{AT}(t) = P_{AA}(t)\mu = \mu \exp(-(\mu+\lambda)t))$$

Solving this

$$P_{AT}(t) = \left[\frac{-\mu}{(\mu+\lambda)} \exp(-(\mu+\lambda)t)\right]_0^T = \frac{\mu}{\mu+\lambda}(1-\exp(-(\mu+\lambda)T))$$

Claims become $\frac{\mu}{\mu+\lambda} C(1-\exp(-(\mu+\lambda)T))$

Question 8

i. $\{0, 1, 2, 3, 4 \ldots\}$

ii.

$$0 \underset{\mu}{\longleftarrow} 1 \underset{\mu}{\overset{\lambda}{\rightleftarrows}} 2 \underset{\mu}{\overset{\lambda}{\rightleftarrows}} 3 \underset{\mu}{\overset{\lambda}{\rightleftarrows}} 4 \underset{\mu}{\overset{\lambda}{\rightleftarrows}} \underset{\mu}{\overset{\lambda}{\cdots}}$$

iii. Generator Matrix:

$$\begin{array}{l c c c c c c}
\text{Lives} & 0 & 1 & 2 & 3 & 4 & \cdots \\
 & \multicolumn{5}{c}{\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
\mu & -(\mu+\lambda) & \lambda & 0 & 0 \\
0 & \mu & -(\mu+\lambda) & \lambda & 0 \\
0 & 0 & \mu & -(\mu+\lambda) & \lambda \\
0 & 0 & 0 & \mu & -(\mu+\lambda)
\end{pmatrix}} &
\end{array}$$

iv. A Markov Jump Process X_t is examined only at the times of transition, the resulting process is called the Jump chain associated with X_t.

v.

$$\begin{array}{l c c c c c c}
\text{Lives} & 0 & 1 & 2 & 3 & 4 & \cdots \\
 & \multicolumn{6}{c}{\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \text{etc} \\
\mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 & 0 & \\
0 & \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 & \\
0 & 0 & \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & \\
0 & 0 & 0 & \mu/(\mu+\lambda) & 0 & \\
\text{etc.} & & & & &
\end{pmatrix}}
\end{array}$$

vi. $\left(\dfrac{\mu}{\mu+\lambda}\right)^3$

Question 10

i. $\frac{d}{dt} P_{\bar{A}\bar{A}}(t) = -2t \times P_{\bar{A}\bar{A}}(t)$

$\frac{d}{dt}\left[\ln P_{\bar{A}\bar{A}}(t)\right] = -2t$

$\ln P_{\bar{A}\bar{A}}(s) = -s^2 + \text{constant}$

$P_{AA'}(0) = 1$, constant $= 0$

$\therefore P_{\bar{A}\bar{A}}(s) = \exp(-s^2)$

ii. P (1^{st} visit to B at time T in state A at $t \neq 0$)

$= \int_0^T P(\text{remains in A to time } s)$

$\times$ P (transition to B in time s, s+ds)
$\times$ P (remains in B to time T) ds

$= \int_{s=0}^{T} P_{\bar{A}\bar{A}}(s) \times 2s \times P_{\bar{B}\bar{B}}(s, T)\, ds$

$= \int_{s=0}^{T} e^{-s^2} \times 2s \times e^{-T^2+s^2}\, ds.$

$= \int_{s=0}^{T} 2s \times e^{-T^2+s^2}\, ds$

$= \int_{s=0}^{T} 2s \times e^{-T^2}\, ds$

$= e^{-T^2} \times T^2$

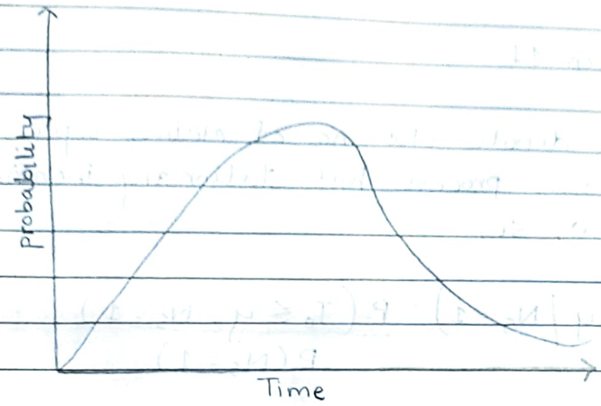
iii.
a.

b.

c.

iii.

a.



b. Initially probability increases from 0 at T=0 and accelerates as the transition rate from A to B increases. However, as transitions increase, it becomes more likely that the process has already visited state B and jumped back to A. Therefore, the probability of being in the 1st visit to B tends (exponentially) to 0.

c. Differentiate to find turning point.

$$\frac{d}{dt}\left[e^{-t^2} \times t^2\right] = 2t \times e^{-t^2} - 2t^3 \times e^{-t^2}$$, equate to 0.

$$e^{-t^2} \times 2t \times (1-t^2) = 0$$

This is maxima

Question 11

i. Let N_t denote the no. of claims upto time t. Since the Poisson process has stationary increments, we may take $t=0$, so,

$$P(T_0 \le y \mid N_s = 1) = \frac{P(T_0 \le y, N_s = 1)}{P(N_s = 1)}$$

$$= \frac{P(N_y = 1, N_s - N_y = 0)}{P(N_s = 1)}$$

$N_s - N_y$ is independent of N_y and has the same distribution as N_{s-y}

RHS =

$$\frac{(\lambda y e^{-\lambda y}) e^{-\lambda(s-y)}}{\lambda s e^{-\lambda s}} = \frac{y}{s}$$

which is the CDF of uniform distribution on $[0,s]$

ii. $\because$ Holding times are independent and have an exponential distribution.

Joint Density $\rightarrow$ $\lambda^n e^{-\lambda(t_1+t_2+t_3 \ldots t_n)} 1_{\{t_1, t_2 \ldots t_n > 0\}}$

iii $P(N_s = k / N_t = n)$

$$= \frac{P(N_s = k, N_t = n)}{P(N_t = n)}$$

$$= \frac{P(N_s = k, N_t - N_s = n-k)}{P(N_t = n)}$$

$$= \frac{\frac{e^{-\lambda s}(\lambda s)^k}{k!} \cdot \frac{e^{-\lambda(t-s)} \lambda^{n-k} (t-s)^{n-k}}{(n-k)!}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{e^{-\lambda t} \lambda^n s^k (t-s)^{n-k}}{k!\,(n-k)!} \cdot \frac{n!}{e^{-\lambda t} \lambda^n t^n}$$

$$= \frac{n!}{k!\,(n-k)!} \cdot \frac{s^k (t-s)^{n-k}}{t^k\, t^{n-k}}$$

$$= \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}$$

which is binomial with parameters n and $\frac{s}{t}$.