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Probability and Statistics - 2

Q.1] Let X_i be claim size on home insurance and Y_i be claim size on car insurance.

So, as given,

$$X \sim N(800, 100^2)$$
$$Y \sim N(1200, 300^2)$$

$$P(Y_1 + Y_2 + Y_3 \cdots > X_1 + X_2 + X_3 + X_4 + 800).$$
$$P[(Y_1 + Y_2 + Y_3 - (X_1 + X_2 + X_3 + X_4) > 800]$$

$$(Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) \sim N(3 \times 1200 - 4(800), 300^2(3) + 100^2)$$

$$\sim N(400, 310000)$$

$$P\left(Z > \frac{800 - 400}{\sqrt{310,000}}\right)$$

$$P(Z > 0.71842) \Rightarrow 1 - P(Z < 0.71842)$$

$$\Rightarrow 1 - P(Z < 0.72)$$
$$= 0.23576.$$

$\therefore$ Probababiliy that after the next 4 home claims and 3 car claims the total size of car claims exceeds the total size of home claims is 0.23576.

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Q.2.] No. of claims $\sim N$

$P(N=n) = {}^{n+2}C_n \, 0.9^3 \, 0.1^n \quad -(1)$

P.d.f of Negative Binomial distribution,

$P(X=x) = {}^{\cancel{x}\,x-1}C_{K-1} \, p^K \, q^{x-K} \quad -(2)$

On comparing them i.e eq. (1) and (2)

$n = K-1 \quad -(3)$

$n+2 = x-1 \quad -(4)$

moreover

$\boxed{K=3}$ and $\boxed{p=0.9}$

$\therefore$ putting $K=3$ in eq. (3)

$n = 3-1 = 2 \quad \boxed{n=2}$

Similarly substitute $n=2$ in eq. (4)

$4 = x-1$

$\therefore \boxed{x=5}$

$\therefore N \sim \text{Neg Bin}(\underset{p}{0.9}, \underset{K}{3})$

Let $X_i \sim$ claim amounts.

$\therefore X_i \sim \text{Gamma}(\underset{\alpha}{6}, \underset{\lambda}{2})$

Y – total claim amount.

$\therefore$ ~~Let~~ MGF of $Y = M_Y(t) = E(e^{tY})$

$= E(e^{tX_1+tX_2+tX_3 \cdots})$

$= E(e^{tX_1}) \cdot E(e^{tX_2}) \cdot E(e^{tX_3}) \cdots$

$= M_{X_1}(t) \cdot M_{X_2}(t) \cdot M_{X_3}(t) \cdots$

$= [M_{X_1}(t)]^n$

$= \left(1 - \frac{t}{2}\right)^{-6n}$

ii) Standard deviation of total claim amount $= \sqrt{\text{Variance}}$.

$$V(S) = V\left[E\left(S/N\right)\right] + E\left[V\left(S/N\right)\right]$$

$$= V\left[E\left(X_1+X_2+X_3 \cdots X_N/N\right)\right] + E\left[V\left(X_1+X_2+X_3+X_4/N\right)\right]$$

$$= V\left[n \cdot E(X)\right] + E\left[V(X)\cdot n\right]$$

$$= \left[E(X)\right]^2 \cdot V(N) + V(X)\cdot E(N)$$

$$= \left(\frac{6\times 6}{2 \cdot 2}\right) \cdot \frac{3}{0.9\times 0.9} + \frac{6}{4} \cdot \frac{\cancel{0.}3}{0.9}$$

$$= \frac{6}{4}\cdot\frac{3}{0.9}\left[2\cancel{6}\times \frac{1}{\cancel{0.9}\ 0.3} + 1\right]$$

$$= \frac{6}{4}\cdot\frac{3}{0.9}\cdot\frac{2.3}{0.3}$$

$$= \frac{\cancel{6}}{\cancel{4}}\cdot\frac{\cancel{30}}{3\cancel{9}}\cdot\frac{23}{\cancel{3}}$$

$$= 38.3333$$

$\therefore$ Standard deviation $= 6.191391873.$

9.3

$$f_X(x) = \lambda e^{-\lambda(x-\alpha)} \qquad x \geq \alpha \text{ and } \lambda, \alpha > 0$$

$$MGF = \cancel{\lambda\int_\alpha^\infty e} \quad E(e^{tx}) = \int_\alpha^\infty e^{tx}\cdot\lambda e^{-\lambda(x-\alpha)}\, dx$$

$$\therefore M_X(t) = \lambda\int_\alpha^\infty e^{tx}\cdot e^{-\lambda x}\cdot e^{\lambda\alpha}\, dx$$

$$= e^{\lambda\alpha}\cdot\lambda\int_\alpha^\infty e^{-(\lambda-t)x}\, dx$$

$$= \frac{e^{\lambda\alpha}\cdot\lambda}{t-\lambda}\left[e^{-(\lambda-t)x}\right]_\alpha^\infty$$

$$= \frac{e^{\lambda\alpha}\lambda}{t-\lambda}\left[-e^{-(\lambda-t)\alpha}\right]$$

$$= \frac{e^{\lambda\alpha}\lambda}{\lambda-t} \cdot e^{-\lambda\alpha} \cdot e^{t\alpha}$$

$$= e^{t\alpha} \frac{\lambda}{\lambda-t}$$

ii) $M_x(t) = \frac{\lambda e^{t\alpha}}{\lambda-t}$

$$= \lambda e^{t\alpha}(\lambda-t)^{-1}$$

$\therefore$ $M'_x(t) = \lambda\left[e^{t\alpha}(\lambda-t)^{-2}(1) + (\lambda-t)^{-1}e^{t\alpha}(\alpha)\right]$

$$= \left[\lambda(\lambda-t)^{-1}e^{t\alpha}(\alpha) + e^{t\alpha}(\lambda-t)^{-2}\right]$$

$$= \lambda e^{t\alpha}(\lambda-t)^{-2}\left[\alpha(\lambda-t)+1\right]$$

put $t=0$

$$= \lambda(\lambda)^{-2}\left[\alpha(\lambda)+1\right]$$

$$= \frac{1}{\lambda}\left[\lambda\alpha+1\right]$$

$$= \frac{\lambda\alpha+1}{\lambda}$$

$M''_x(t) = \Big[\lambda \alpha e^{t\alpha}(\lambda-t)^{-2} + \alpha^2(\lambda-t)^{-1}e^{t\alpha} + e^{t\alpha}(-2)(\lambda-t)^{-3} + (\lambda-t)^{-2}e^{t\alpha}\Big]$

put $t=0$

$M''_x(t) = \lambda\left[\alpha(\lambda)^{-2} + \alpha^2(\lambda)^{-1} + 2(\lambda)^{-3} + 2(\lambda)^{-2}\right]$

$$= \lambda\left[\frac{2\alpha}{\lambda^2} + \frac{\alpha^2}{\lambda} + \frac{2}{\lambda^3}\right]$$

$$E(x^2) = \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2}$$

$$\therefore V(x) = E(x^2) - [E(X)]^2$$

$$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda} - \left[\alpha^2 + \frac{1}{\lambda^2} + \frac{2\alpha}{\lambda}\right] //$$

$$\boxed{\therefore V(x) = \frac{1}{\lambda^2}}$$

Q.4) $G_v(t) = \left(\frac{p}{1-qt}\right)^k$

$$G_v(t) = \Sigma t^v . P(V=v)$$

$$\therefore P(V=4) = \frac{\sqrt{k+4}}{\sqrt{5}.\sqrt{k}} p^k q^k$$

$$= \frac{\sqrt{k+4}}{4!\sqrt{k}} p^k q^k.$$

$$\therefore P(V=4) = \frac{(k+3)!}{(k-1)!\,4!} p^k (1-q)^k$$

Q.5]

$$f_{x,y}(x,y) = ax(x+y) \qquad 0 < x < 5,\ 10 < y < 20.$$

$$\int_x\int_y f_{x,y}(x,y)\,dx\cdot dy = 1.$$

$$\therefore \int_{10}^{20}\int_{0}^{5} ax(x+y)\cdot dx\cdot dy = 1$$

$$\therefore a\int_{10}^{20}\int_{0}^{5} x^2+xy\cdot dx\cdot dy = 1$$

$$\therefore a\int_{10}^{20}\left[\frac{x^3}{3}+\frac{x^2}{2}y\right]_0^5 \cdot dy = 1$$

$$\therefore a\int_{10}^{20}\left[\left(\frac{5^3}{3}+\cancel{x}\frac{5^2}{2}y\right)-\left(\frac{0}{3}+\frac{0y}{2}\right)\right] = 1$$

$$\therefore a\int_{10}^{20}\left[\frac{125}{3}+\frac{25y}{2}\right] = 1$$

$$\therefore a\left[\frac{125}{3}y+\frac{25}{2}\cdot\frac{y^2}{2}\right]_{10}^{20} = 1$$

$$\therefore a\left[\left(\frac{125\times 20}{3}+\frac{25\times\cancel{20}^{10}\times\cancel{20}^{10}}{\cancel{2}\times\cancel{2}}\right)-\left(\frac{125\times 10}{3}+\frac{25\times\cancel{10}^{5}\times\cancel{10}^{5}}{\cancel{2}\times\cancel{2}}\right)\right] = 1$$

$$\therefore a\left[\left(\frac{2500}{3}+2500\right)-\left(\frac{1250}{3}+625\right)\right] = 1$$

$$\therefore a\left(\frac{10{,}000}{3}-\frac{3125}{3}\right) = 1$$

$$\therefore a = \frac{3}{6875}$$

$P\left(Y > \frac{1}{3}X + 10\right)$

$E\left(Y/X\right) =$

ii) $\cancel{E\left(Y/x\right) = \int_{0}^{5} \frac{3}{68775}\, x \cdot f_{x/y}}$

$$E\left(Y/_x\right) = \int_{10}^{20} y \cdot f_{Y/_X}(y/x)\, dy$$

$$= \int_{10}^{20} y \cdot \frac{f_{x,y}(x,y)}{f_x(x)}\, dy.$$

Q.6] $X \sim Poi(\lambda)$

$Y \sim Poi(\mu)$

To Prove $(X-Y) \sim Poi(\lambda-\mu)$ & $(X+Y) \sim Poi(\lambda+\mu)$

$M_X(t) = e^{\lambda(e^t-1)}$

$M_Y(t) = e^{\mu(e^t-1)}$

let $Z = X - Y$ and $Q = X + Y$

$\therefore M_z(t) = E(e^{tz})$

$= E[e^{t(x-y)}]$

$= E(e^{tx} \, e^{-ty})$

$= E(e^{tx}) \cdot E(e^{-(ty)})$

$= e^{\lambda(e^t-1)} \cdot e^{-\mu(e^t-1)}$

$= e^{(\lambda-\mu)(e^t-1)}$

$\therefore$ Z i.e $X-Y \sim Poi(\lambda-\mu)$.

$Q = X+Y$

$\therefore M_Q(t) = E[e^{tq}]$

$= E[e^{t(x+y)}]$

$= E[e^{tx} \cdot e^{ty}]$

$= E(e^{tx}) \cdot E(e^{ty})$

$= M_X(t) \cdot M_Y(t)$

$= e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)}$

$= e^{\lambda(e^t-1)+\mu(e^t-1)}$

$= e^{(\lambda+\mu)(e^t-1)}$

$\therefore$ Q i.e $X+Y \sim Poi(\lambda+\mu)$

Q.7] i) $Var\left(X/Y=2\right) = E\left(X^2/Y=2\right) - \left[E\left(X/Y=2\right)\right]^2$

$$E\left(X^2/Y=2\right) = \sum x^2 \cdot \frac{P(X=x, Y=2)}{P(Y=2)}$$

$$\therefore E\left(X^2/_{Y=2}\right) = 1^2 \cdot \frac{0}{0.6} + 2^2 \cdot \frac{0.3}{0.6} + 3^2 \cdot \frac{0.1}{0.6} + 4^2 \cdot \frac{0.2}{0.6}$$

$$= 1 \times 0 + 4 \times \frac{1}{2} + 9 \cdot \frac{1}{6} + 16 \cdot \frac{2}{6}$$

$$= 0 + 2 + \frac{3}{2} + \frac{16}{3}$$

$$= 8.8333$$

$$E\left(X/_{Y=2}\right) = \sum x \cdot \frac{P(X=x, Y=2)}{P(Y=2)}$$

$$= \frac{1 \times 0}{0.6} + 2 \times \frac{0.3}{0.6} + 3 \times \frac{0.1}{0.6} + 4 \times \frac{0.2}{0.6}$$

$$= 0 + 2 \times \frac{1}{2} + 3 \times \frac{1}{6} + 4 \times \frac{1}{3}$$

$$= 1 + \frac{1}{2} + \frac{4}{3}$$

$$= 2.8333$$

$$\therefore V\left(X/_{Y=2}\right) = 8.83333 - (2.8333)^2$$

$$= 8.8333 - (8.02777)$$

$$= 0.30523$$

$$\boxed{\therefore V\left(X/_{Y=2}\right) = 0.30523}$$

ii) $f_{U,V}(u,v) = \frac{48}{67}(2uv - u^2) \quad 0 < u < 1, \ \frac{u}{2} < v < 2$

$$E\left(U/_{V=v}\right) = \int_0^1 u \cdot f_{U/V}(u/v) \cdot du = \int_0^1 u \cdot \frac{f_{U,V}(u,v)}{f_V(v)} \cdot du$$

$$f_V(v) = \int_0^1 f_{U,V}(u,v)\,du$$

$$= \int_0^1 \frac{48}{67}(2uv - u^2)\,du$$

$$= \frac{48}{67}\int_0^1 2uv - u^2\,du$$

$$= \frac{48}{67}\left[\cancel{2}v\left[\frac{u^2}{\cancel{2}}\right]_0^1 - \left(\frac{u^3}{3}\right)_0^1\right]$$

$$= \frac{48}{67}\left[v(1-0) - \left(\frac{1-0}{3}\right)\right]$$

$$= \frac{48}{67}v - \frac{\cancelto{16}{48}}{67}\times\frac{1}{\cancel{3}}$$

$$= \frac{48}{67}v - \frac{16}{67}$$

$$= \frac{48v-16}{67}$$

$$\therefore E\left(U/V=v\right) = \int_0^1 u\cdot\frac{\frac{48(2uv-u^2)}{67}}{\frac{48v-16}{67}}\,du$$

$$= \frac{67}{48v-16}\cdot\frac{48}{67}\int_0^1 u(2uv-u^2)\,du$$

$$= \frac{48}{48v-16}\int_0^1 2u^2v - u^3\,du$$

$$= \frac{3\cdot\cancel{48}}{\cancelto{1}{16}(3v-1)}\left[2\left(\frac{u^3}{3}\right)_0^1 v - \left(\frac{u^4}{4}\right)_0^1\right]$$

$$= \frac{3}{3v-1}\left[2v\left(\frac{1-0}{3}\right) - \left(\frac{1-0}{4}\right)\right]$$

$$= \frac{3}{3v-1}\left(\frac{2v}{3} - \frac{1}{4}\right)$$

$$= \frac{\cancel{3}}{3v-1} \times \frac{8v-3}{\cancel{12}\ 4}$$

$$= \frac{8v-3}{12v-1}$$

$$\boxed{\therefore E\left(\frac{V}{V=v}\right) = \frac{8v-3}{12v-1}}$$

Q.8] $X \sim \text{Gamma}(\underset{\alpha}{3}, \underset{\lambda}{2})$

$$E\left(\frac{Y}{X=x}\right) = 3x+1 \qquad V\left(\frac{Y}{X=x}\right) = 2x^2+5$$

$$E(Y) = E\left[E\left(\frac{Y}{X=x}\right)\right]$$

$$= E[3x+1]$$

$$= 3 \cdot E[x] + 1$$

$$= 3 \times \frac{3}{2} + 1$$

$$= \frac{9}{2} + 1$$

$$= \frac{11}{2}.$$

$$\boxed{\therefore E(Y) = \frac{11}{2}}$$

$$V(Y) = E\left[V\left(\frac{Y}{X=x}\right)\right] + V\left[E\left(\frac{Y}{X=x}\right)\right]$$

$$= E(2x^2+5) + V(3x+1)$$

$$= 2 \cdot E[X^2] + 5 + 3^2 \cdot V(X)$$

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$E[X^2]$ can be found by differentiating twice the M.G.F of X.

$= (2\times3+5) + 9\times\frac{3}{4}$

$= 11 + \frac{27}{4}$

$= \frac{71}{4}$

$\therefore \boxed{V(Y) = \frac{71}{4}}$

Q.9] X $\mu = 3$ $\sigma = 2$
Y $\mu = 4$ $\sigma = 1$
$Corr(X,Y) = -0.3$

$Cov(X,Y) = Corr(X,Y) \times \sigma_x \cdot \sigma_y$

$Z = X+Y$

$= -0.3\times(2\times1)$

$= -0.6$

$Cov(X,Z) = Cov(X, X+Y)$
$= Cov(X,X) + Cov(X,Y)$
$= V(X) + Cov(X,Y)$
$= 4 + (-0.6)$
$= 4 - 0.6$
$= 3.4.$

$Var(Z) = Var(X+Y)$
$= Var(X) + Var(Y) + 2 Cov(X,Y)$
$= 4 + 1 + 2\times -0.6$
$= 5 + (-1.2)$
$= 3.8$

Q.10]

$$f_{x,y}(x,y) = kx^{-\alpha} \cdot e^{-y/\beta} \qquad 1 < x < \infty, \; 1 < y < \infty$$

$$\int_1^\infty \int_1^\infty f_{x,y}(x,y) \cdot dx\, dy = 1.$$

$$\therefore \; k \int_1^\infty \int_1^\infty x^{-\alpha} \cdot e^{-y/\beta} \cdot dx \cdot dy = 1$$

$$\therefore \; K \int_1^\infty \left(\frac{x^{-\alpha+1}}{-\alpha+1} \right)_1^\infty \cdot e^{-y/\beta} \cdot dy = 1$$

$$\therefore \; K \int_1^\infty \left[0 - \frac{1}{-(\alpha+1)} \right] \cdot e^{-y/\beta} \cdot dy = 1$$

$$\therefore \; K \cdot \frac{1}{\alpha+1} \cdot \int_1^\infty e^{-y/\beta} \cdot dy = 1$$

$$\therefore \; K \cdot \frac{1}{\alpha+1} \left[\frac{e^{-y/\beta}}{-1/\beta} \right]_1^\infty = 1$$

$$\therefore \; K \cdot \frac{1}{\alpha+1} \left[0 - \frac{e^{-1/\beta}}{-1/\beta} \right] = 1$$

$$\therefore \; \frac{K}{\alpha+1} \cdot \beta \cdot e^{-1/\beta} = 1$$

$$\boxed{\therefore \; K = \frac{\alpha+1}{\beta \cdot e^{-1/\beta}}}$$