—AMAN. TRIPATHI
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Statistical & Risk Modeling - 4
ASSIGNMENT

(1) Y_t follows the process:

$$Y_t^2 - \beta_1 e_t^2 Y_{t-1}^2 = 2(Y_t - \beta_1 e_t^2 Y_{t-1})\mu - (1-\beta_1 e_t^2)\mu^2 + \beta_0 e_t^2$$

(i) $\Rightarrow Y_t = \mu + e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5}$

$\Rightarrow E(Y_t) = E(\mu) + E(e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5})$

So e_t & Y_{t-1} are independent

$\Rightarrow E(Y_t) = \mu + E(e_t)E((\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5})$

$\Rightarrow E(Y_t) = \mu + 0 \times E((\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5})$

$\therefore E(Y_t) = \mu$

Next, $Cov(Y_t, Y_{t-s}) = E(Y_t Y_{t-s}) - E(Y_t)E(Y_{t-s})$

$\Rightarrow Cov(Y_t, Y_{t-s}) = E((\mu + e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5})(\mu + e_{t-s}(\beta_0 + \beta_1(Y_{t-s-1} - \mu)^2)^{0.5})) - \mu^2$

$\Rightarrow Cov(Y_t, Y_{t-s}) = E(\mu^2 + \mu e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5} + \mu e_{t-s}(\beta_0 + \beta_1(Y_{t-s-1} - \mu)^2)^{0.5}$
$+ e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5} e_{t-s}(\beta_0 + \beta_1(Y_{t-s-1} - \mu)^2)^{0.5}) - \mu^2$

$\Rightarrow Cov(Y_t, Y_{t-s}) = \mu^2 + 0 + 0 + 0 \times E((\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5}(\beta_0 + \beta_1(Y_{t-s-1} - \mu)^2)^{0.5}) - \mu^2$

$\Rightarrow Cov(Y_t, Y_{t-s}) = 0$

(ii) $Var(Y_t | Y_{t-1}) = Var(e_t) Var(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)$
$= \beta_0 + \beta_1(Y_{t-1} - \mu^2)$

We can see that Y_t depends on Y_{t-1} in terms of variance. They are dependent

(iii) 1st difference: $\Delta X_t = X_t - X_{t-1} \Rightarrow E(\Delta X_t) = E(X_t) - E(X_{t-1})$

$\Rightarrow E(\Delta X_t) = E(0.5Y_t + 0.3t + 0.1) - E(0.5Y_{t-1} + 0.3(t-1) + 0.1)$

$\Rightarrow E(\Delta X_t) = 0.5\mu + 0.3t + 0.1 - 0.5\mu - 0.3(t-1) - 0.1 = 0.3$

$\Rightarrow$ Mean is independent of t & hence constant.

$Cov(\Delta X_t, \Delta X_{t-s}) = Cov(X_t - X_{t-1}, X_{t-s} - X_{t-s-1})$

$\Rightarrow Cov(\Delta X_t, \Delta X_{t+s}) = Cov(0.3 + Y_t - Y_{t-1}, 0.3 + Y_{t+s} - Y_{t+s-1})$

$\Rightarrow Cov(\Delta X_t, \Delta X_{t+s}) = 0 - 0 - 0 + 0 = 0$.

The auto covariance fn. is constant $\therefore$ the 1st diff. of X_t is stationary

(2) $Y_t = Y_{t-1} + 0.5 Y_{t-2} - 0.5 Y_{t-3} + Z_t + 0.3 Z_{t-1}$

(i) Characteristic eqn: $1 - z - 0.5z^2 + 0.5z^3 = 0$

$= (1-z)(1 - 0.5z^2)$

Rewriting model the model in terms of $x = (1-B)Y$

$\Rightarrow X_t - 0.5X_{t-2} = Z_t + 0.3Z_{t-1}$

This is ARMA (2,1)

$\Rightarrow Y_t$ is ARIMA (2,1,1)

(ii) characteristic polynomial of X_t is $(1 - 0.5z^2)$ whose roots are $\pm\sqrt{2}$.

X_t is stationary $\therefore$

(iii) model equation is $X_t - 0.5 X_{t-2} + Z_t + 0.3 Z_{t-1}$

take covariances of both sides

ACF of $X_t \Rightarrow \rho(k) = \begin{cases} (0.5)^{|k|/2} & \text{if } k \text{ is even} \\ (1.5/109)(0.5)^{(|k|-1)/2} & \text{if } k \text{ is odd} \end{cases}$

(3)(i) $Y_t = 0.4Y_{t-1} + 0.2Y_{t-2} + Z_t + 0.025 Z_{t-1} + 0.016$

(a) $\Rightarrow (1 - 0.4B - 0.2B^2) Y_t = (1 + 0.025B) Z_t + 0.016$

Characteristic equation: $1 - 0.4z - 0.2z^2 = 0$

$\rightarrow$ no root w/ magnitude 1, $\therefore d = 0$

$\therefore$ the process is ARIMA (2,0,1)

(b) $E(Y_t) = 0.016 / 0.4 = 0.04 = 4\%$ $[(1 - 0.4 - 0.2)E(Y_t) = 0.016$

(c) The 2 roots of the characteristics eqn are $-1 \pm \sqrt{6}$ i.e. 1.4495 and -3.4495 (magnitude > 1) $\therefore$ the process is stationary

(ii) $X_t = 0.4X_{t-1} + 0.2X_{t-2} + Z_t + 0.025$

ACF for the 1st few lags are $\rho_1 = 0.5, \rho_2 = 0.4, \rho_3 = 0.26, \rho_4 = 0.184$

b) . Diagnostic checks:

- inspection of the graph of the time-plot of residuals: Visual inspection might reveal a pattern such as uneven fluctuations or clusters of only +ve/only -ve negative residuals, which indicate inadequate fit
- Inspection of the sample ACFs of the residuals: Too many ACF or PACF values outside the range $\pm 2/\sqrt{N}$ may indicate poor fit or too few parameters

(4) AR(2) process: $Y_t = -\alpha Y_{t-1} + \alpha^2 Y_{t-2} + Z_t$

(i) model can be written as: $(1+\alpha B - \alpha^2 B^2) Y_t = Z_t$

$\therefore$ characteristic eqn is: $1 + \alpha x - \alpha^2 x^2 = 0$

applying quadratic formula, $x = (1 \pm \sqrt{5})\, 2a$

$\Rightarrow$ roots lie outside unit circle $\therefore$ we require that

$(\sqrt{5}-1)/2\,|a| > 1$ & $(\sqrt{5}+1)/2 \cdot |\alpha| > 1 \Rightarrow |a| < (\sqrt{5}-1)/2$

(ii) $Y_t = -\alpha Y_{t-1} + \alpha^2 Y_{t-2} + Z_t$

Yule Walker equations are:

$Cov(Y_t, Y_t) = \gamma_0 = -\alpha \gamma_1 + \alpha^2 \gamma + \sigma^2$ — (1)

$Cov(Y_t, Y_{t-1}) = \gamma_1 = -\alpha \gamma_0 + \alpha^2 \gamma_1$ — (2)

$Cov(Y_t, Y_{t-2}) = \gamma_2 = -\alpha \gamma_1 + \alpha^2 \gamma_0$ — (3)

from (2), $\gamma_1 = -a\gamma_0 / (1-\alpha^2)$ — (4)

substituting γ_1 in (3) we get (5)

$\gamma_2 = -\alpha[-a\gamma_0 / (1-\alpha^2)] + \alpha^2 \gamma_0 = (2\alpha^2 - \alpha^4)\gamma_0/(1-\alpha^2)$

substituting γ_1 and γ_2 into (1) we have

$\gamma_0 = -\alpha(-\alpha\gamma_0/(1-\alpha^2)) + \alpha^2(2\alpha^2 - \alpha^4)\cdot\gamma_0/(1-\alpha^2) + \sigma^2$

$\therefore \gamma_0 = \sigma^2(1-\alpha^2)/(1-2\alpha^2 - 2\alpha^4 + \alpha^6)$

$\Rightarrow$

$\gamma_1 = -\sigma^2 \alpha / (1 - 2\alpha^2 - 2a^4 + a^6)$

$\gamma_2 = \sigma^2 (2\alpha^2 - \alpha^4)/(1 - 2\alpha^2 - 2\alpha^4 + \alpha^6)$

587.83

(5) $\sum_{i=2}^{200} (x_i - x_{i-1})^2 = 936.49$, $\sum_{i=2}^{199} (x_i - x_{i-1})(x_{i+1} - x_i) =$

ARMA $\Rightarrow (1,1,0)$

(i) model is $(X_n - X_{n-1}) = \alpha (X_{n-1} - X_{n-2}) + \varepsilon_n$, ε_n's are not correlated w/ mean 0 & var σ^2

Since mean is known to be 0, the usual estimator of the parameter α is:

$$\hat{\alpha} = \hat{\rho}_1 = \frac{587.83}{936.49} = 0.6277$$

estimator of autocovariance at lag 0 is

$$\hat{\gamma}_0 = \frac{936.49}{200} = 4.6825$$

$$\hat{\sigma}^2 = (1 - 0.6277^2)(4.6825)$$
$$= 2.8376$$

(ii) forecast of x_{201} is obtained from

$\Rightarrow 0.6277 \times (1.93 - 0.82) = 0.6967$

$\Rightarrow \hat{x}_{200} = 1.93 + 0.6967 = 2.6267$

(6) (i)
- to construct a model to fit data
- for description of data
- for forecasting future values of the process
- deciding whether the process is out of control, & regulation

(ii) MA(1) $\rightarrow$ not markov

AR(1) $\rightarrow$ Markov

AR(2) $\rightarrow$ not Markov

random walk $\rightarrow$ Markov

(iii) $X(t) = a + bt + Y(t)$

(a): $E[X(t)] = a + bt + E[Y(t)]$

Since $Y(t)$ is stationary, $E[Y(t)]$ is equal to some constant c that does not depend on t

$\therefore$ $E[X(t)] = a + bt + c$

$\Rightarrow X(t)$ cannot be stationary

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(iii) $E(\nabla X(t)) = E(X(t)) - E(X(t-1)) = a + bt + c - (a + b(t-1) + c) = b.$

$\therefore$ mean doesn't depend on t

$Cov(\nabla(X(t)), \nabla X(s)) = C_Y(t-s) - C_Y(t-s+1) - C_Y(t-s-1) + C_Y(t-s)$

where C_Y is the covariance function of the stationary process $Y(t)$. The last expression depends on t & s only through the difference $t-s$. Hence $\nabla X(t)$ is stationary.

As $\nabla X(t)$ is stationary but $X(t)$ is not, the process $X(t)$ is $I(1)$

iv) $X_1(t) - X_2(t) = Y(t) = \varepsilon e(t) - e_1(t) - e_2(t)$

$e(t)$ is a white noise process.

$Y(t)$ is stationary as long as $|a-b| < 1$ & is an AR(1) process.

$\Rightarrow$ processes are cointegrated whenever $|a-b| < 1$

(7) (i) • Auto regressive process (AR)

→ an AR process of order p is a sequence of random variables (X_t) defined consecutively by the rule:

$X_t = \mu + \alpha_1 (X_{t-1} - \mu) + \alpha_2 (X_{t-2} - \mu) + \dots + \alpha_p (X_{t-p} - \mu) + e_t$

Thus the AR model attempts to explain the current value of X as a linear combination of past values w/ some additional externally generated random variables.

• Moving average process (MA)

$X_t = \mu + e_t + \beta_1 e_{t-1} + \dots + \beta_q e_{t-q}$

• ARMA

> combines AR & MA processes.

(ii) a) ARIMA (0,0,1)

b) ARMA (2,3) $\Rightarrow$ ~~AR~~

c) ARIMA (1,1,1)

(iii) $\rho_0 = 1$; $\rho_{\pm 1} = 0.364$; $\rho_{\pm 2} = 0.409$; $\rho_{\pm 3} = 0.182$

$\rho_{\pm K} = 0$ for $|K| > 3$.

(5) (i) $X_t = A_t + S_t$

$A_t = 0.5 A_{t-1} + 0.5 B_{t-1} + e_t^{(A)}$

$B_t = 0.7 A_{t-1} - 0.7 A_{t-2} + e_t^{(B)}$

$$Y_t = \begin{pmatrix} A_t \\ B_t \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.7 & 0 \end{pmatrix} \begin{pmatrix} A_{t-1} \\ B_{t-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -0.7 & 0 \end{pmatrix} \begin{pmatrix} A_{t-2} \\ B_{t-2} \end{pmatrix} + \begin{pmatrix} e_t^{(A)} \\ e_t^{(B)} \end{pmatrix}$$

The Eigen values of first matrix are given by the following equation

$\lambda^2 - 0.5\lambda - 0.35 = 0$

Hence, $|\lambda| < 1$

similarly for 2nd matrix, $|\lambda| < 1$

as all the eigen values are less than 1, the process Y_t is stationary

(ii) a) X_t is ARIMA (2,1,0) process with $|\alpha| < 2$

b) $\lambda_2 = \frac{1}{0.5} r_1 - \lambda_1 = \frac{\sigma^2}{(1 - 0.25\alpha^2)^2}$

$\Rightarrow y_{s2} = 0.04 y_{s1} - 0.0004 (x_{50} - x_{49})$

(9) (i) $e_{2t} = \sigma_e \sqrt{-2 \log U_{2t}} \sin(2\pi U_{2t+1})$ → $\sim$ random U_{2t}

(a) $e_{2t+1} = \sigma_e \sqrt{-2 \log U_{2t}} \cos(2\pi U_{2t+1})$ → $N(0, \sigma_e^2)$ → feed e_t into formula

(b) ability to reuse pseudo random number sequence is important when comparing ability of different mechanisms

(ii) The models do not possess correct correlation structure

(iii) a) $\rho_1 = Corr(X_t, X_{t-1}) = \alpha_1 corr(X_{t-1}, X_{t-1}) + \alpha_2 Corr(X_{t-2}, X_{t-1})$

$= \alpha_1 + \alpha_2 \rho_1$

$\Rightarrow \rho_1 = \alpha_1 / (1 - \alpha_2)$

$\rho_2 = Corr(X_t, X_{t-2}) = \alpha_1 Corr(X_{t-1}, X_{t-2}) + \alpha_2 Corr(X_{t-2}, X_{t-2})$

$= \alpha_1 \rho_1 + \alpha_2$

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b) $0.7 = \rho_1 = \alpha_1 / (1-\alpha_2)$

$\Rightarrow \alpha_1 = \frac{35}{51}$ & $\alpha_2 = \frac{1}{51}$

(10) i) $p=1$, $q=4$ hence follows ARMA (1,4)

ii) 4 Non-linear or non-stationary time series models:

→ Bilinear models show 'bursty' behaviour

$$X_n - \alpha(X_{n-1} - \mu) = \mu + e_n + \beta e_{n-1} + b(X_{n-1} - \mu)e_{n-1}$$

→ Threshold autoregressive models show 'cyclical' behaviour

$$X_n = \begin{cases} \mu + \alpha_1 (X_{n-1} - \mu) + e_n & \text{if } X_{n-1} \le d \\ \alpha_2 (X_{n-1} - \mu) + e_n, & \text{if } X_{n-1} > d \end{cases}$$

→ Random coefficient, autoregressive models are a sequence of independent random variables

$X_t = \mu + \alpha_1 (X_{t-1} - \mu) + e_t$ where $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ are independent random variables

→ ARCH models: used to model asset prices where volatility depends on size of previous value

$$X_t = \mu + e_t \sqrt{\alpha_0 + \sum_{k=1}^{p} \alpha_k (X_{t-k} - \mu)^2}$$

iii) $X_t \sim MA(1)$

$\Rightarrow X_t = e_t + \beta e_{t-1}$, where $e_t \sim (0, \sigma^2)$

⋮

std. dev. of 1st diff is higher of Y_t than that of X_t

(11) i) used for description of data, to construct a model which fits the data, forecasting future values of process & deciding if process is out of control & requires action

ii) univariate time series is a sequence of observations recorded at regular intervals. State space is continuous but time set is discrete

a time series is invertible if we can write the white noise terms as a convergent sum of the X terms

a time series process X has the markov property if we can predict the future state at time t from the current state alone.

iii) We can use method of moving average & seasonal differencing for removing seasonal variation.
& least squares trend removal & differencing for removing linear trends

iv) a) $Y_t = 1.25 + 0.2e_t + 0.19\, e_{t-1} + 1.192 \sum_{i=2}^{\infty} 0.6^{i-2} e_{t-i}$

b) $\rho_0 = 1$, $\rho_1 = 0.828$, $\rho_2 = 0.5323$, $\rho_k = 0.6 \rho_{k-1}$
for $k \geq 3$

(12) i) Lack of stationarity may be caused by the presence of deterministic effects in the quantity being observed.
for example, deterministic trend or cycle such as seasonal effects.
If the process observed is the integrated ~~process~~ version of a more fundamental process

ii)
a) $(1-B)^d \phi B (X_t - \mu) = \theta (B) e_t$ where $\phi(B) = 1 - \sum_1^p B^i \alpha_i$
and $\theta(B) = \sum_1^q B^j \beta_j$

$(\nabla^d X_t - \mu) = \sum_1^p \alpha_i (\nabla^d X_{t-1} - \mu) + e_t + \sum_1^q \beta_j e_{t-j}$

b) Main steps in Box-Jenkins:

- → identification of model from ARIMA class
- → estimation of the parameter in the identified model
- → Diagnostic checks

c) If the sample autocorrelation coefficients decay slowly from 1 then this indicates that further differencing is required. Not the case for $d=1$ which means that the differencing of the original series is required once hence $d=1$

→ correct value of d minimise the sample variance hence $d=1$

d) 1) MA(1) process & stationary

$\Rightarrow$ ARIMA (0, 0, 1)

2) ARMA (2,3) → cannot be differenced

$\Rightarrow$ ARIMA (2, 0, 3)

→ ARMA (2,3) non stationary process

3) ARIMA (1, 1, 1) → stationary

(13) i) stochastic process is weakly stationary if it has constant mean and the covariance is constant for each fixed lag

ii) MA process is $X_n = Z_n + \beta Z_{n-1}$

mean $= E(X_n) = (1+\beta)\mu$

$\text{Var}(X_n) = \text{Var}(Z_n + \beta Z_{n-1})$

$= (1+\beta^2)\sigma^2$

covariance: $\text{Cov}(X_n, X_{n+1}) = \text{Cov}(Z_n + \beta Z_{n-1}, Z_{n+1} + \beta Z_n)$

$= \beta\sigma^2$

for $m > 1$, $\text{Cov}(X_n, X_{n+m}) = \text{Cov}(Z_n + \beta Z_{n-1}, Z_{n+m} + \beta Z_{n+m-1})$

$= 0$ ∴ weakly stationary

iii) $(1 - 1.5B + 0.5B^2)X = Z$

$= (1-B)(1-0.5B)$

Since one of the roots has magnitude 1, process is not stationary

iv) process is ARIMA (1, 1, 0) so $(1-B)X$ is AR(1)

$Y = (1-B)X \quad . \quad (1-0.5B)Y = Z$

autocorrelation at lag 1 is 0.5
