Statistic & Risk modelling assignment.
Unit 3 & 4.

Qu 1.

$$Y_t^2 - \beta_1 e_t^2 Y_{t-1}^2 = 2(Y_t - \beta_1 e_1^2 Y_{t-1})\mu - (1-\beta_1 e_t^2)\mu^2 + \beta_0 e_t$$

(i) $\text{Cov}(Y_t, Y_{t-s})$

$$Y_t^2 - 2Y_t\mu + \mu^2 = e_t^2(\beta_0 + \beta_1 Y_{t-1}^2 - 2\beta_1 Y_{t-1}\mu + \beta_1\mu^2)$$

$$(Y_t - \mu)^2 = e_t^2(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)$$

$$Y_t = \mu + e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{1/2}$$

$$E(Y_t) = E(\mu) + E(e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{1/2})$$

$$E(Y_t) = E(\mu) + 0$$

$$= \mu$$

$$\text{cov}(Y_t, Y_{t-1}) = E(Y_t Y_{t-s}) - E(Y_t)E(Y_{t-s})$$

$$\gamma_s = E(\mu + e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{1/2}$$

$$(\mu + e_{t-s}(\beta_0 + \beta_1(Y_{t-s-1} - \mu)^2)^{1/2} - \mu^2$$

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$= E(\mu^2 + \mu e_t (\beta_0 + \beta_1 (Y_{t-1} - \mu)^2)^{1/2} + \mu e_{t-s} (\beta_0 + \beta_1 \cdots$

$\cdots (Y_{t-s-1} - \mu)^2)^{1/2} + e_t (\beta_0 + \beta_1 (Y_{t-1} - \mu)^2)^{1/2} \times$

$e_{t-s} (\beta_0 + \beta_1 (Y_{t-s-1} - \mu)^2)^{1/2} - \mu^2$

$= \mu^2 - \mu^2 = 0$

(ii) $Var(Y_t \mid Y_{t-1}) = var(e_t)\, var(\beta_0 + \beta_1 (Y_{t-1} - \mu)^2$

$= \beta_0 + \beta_1 (Y_{t-1} - \mu)^2$

Var is dependent on Y_{t-1}

Y_t and Y_{t-s} are independent

(iii) $X_t = 0.5\, Y_t + 0.3t + 0.1$

$E(\Delta X_t) = E(X_t) - E(X_{t-1})$

$= E(0.5\, Y_t + 0.3t + 0.1) - E(0.5\, Y_{t-1} + 0.3(t-1) + 0.1$

$= 0.5\mu + 0.3t + 0.1 - 0.5\mu - 0.3(t-1) - 0.1 = 0.3$

Mean is constant

2.(i) $Y_t = Y_{t-1} + 0.5Y_{t-2} - 0.5Y_{t-3} + Z_t + 0.3Z_{t-1}$

Characteristic eqn

$1 - z - 0.5z^2 + 0.5z^2 = 0$

$X = (1-B)Y$

$X_t - 0.5X_{t-2} = Z_t + 0.3Z_{t-1}$

ARIMA (2, 1)

Y = ARIMA (2, 1, 1)

(ii) $X_t - 0.5X_{t-2}$

$(1 - 0.5z^2)$

Roots $= \pm\sqrt{2}$

> 1

process is stationary

(iii) $X_t = 0.5X_{t-2} + Z_t + 0.3Z_{t-1}$

$Cov(X_t, Z_t) = Cov(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, Z_t)$

$= 0 + \sigma^2 = \sigma^2$

$$\operatorname{cov}(X_t, Z_{t-1}) = \operatorname{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, Z_{t-1})$$

$$= 0 + 0 + 0.3\sigma^2 = 0.3\sigma^2$$

$$\operatorname{cov}(X_t, Z_{t-2}) = \operatorname{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, Z_{t-2})$$

$$= 0 + 0 + 0.5\sigma^2 = 0.5\sigma^2$$

$$\gamma_0 = \operatorname{cov}(X_t, X_t)$$

$$= \operatorname{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, X_t)$$

$$= 0.5\gamma_2 + \sigma^2 + 0.09\sigma^2$$

$$= 0.5\gamma_2 + 1.09\sigma^2 \quad - (1)$$

$$\gamma_1 = \operatorname{cov}(X_t, X_{t-1})$$

$$= \operatorname{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, \cancel{X}_{t-1})$$

$$0.5\gamma_1 + 0 + 0.3\sigma^2 \quad - (2)$$

$$\gamma_k = \operatorname{cov}(X_t, X_{t-k})$$

$$= \operatorname{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, X_{t-k})$$

$$0.5\gamma(k-2) \qquad k > 2$$

$$\gamma_2 = \operatorname{cov}(X_t, X_{t-2}) = \operatorname{cov}(0.5X_{t-2} + Z_t + 0.3Z_{t-1}, X_{t-2})$$

$$= 0.5\gamma_0 \quad - (3)$$

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Replace (3) into (1)

$$\gamma_0 = 0.25\,\gamma_0 + 1.09\sigma^2$$

$$\gamma_0 = \frac{109\sigma^2}{75}$$

$$\gamma_1 = \frac{3\sigma^2}{5}$$

$$\rho_1 = \frac{\gamma_1}{\gamma_0} = \frac{45}{109}$$

From (3) and (4)

$$\rho(k) = 0.5\,\rho_{(k-2)}$$

$\cancel{\rho(k) = 0.5}$

$$Y_t = 0.4Y_{t-1} + 0.2Y_{t-2} + Z_t + 0.025Z_{t-1} + 0.016$$

3. (1) a) $(1 - 0.4B - 0.2B^2)Y_t = (1 + 0.025B)Z_t + 0.016$

Characteristic : $1 - 0.4\lambda - 0.2\lambda^2 = 0$

$$d = 0$$

ARIMA (2, 0, 1) process

(b) $(1 - 0.4 - 0.2)(E(y)) = 0.016$

$$E(Y_t) = \frac{0.016}{0.4} \approx 4\%$$

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(c) Roots : $-1 \pm \sqrt{6}$

1.4495 , -3.4495

>1

Y_t is a stationary process

(ii) (a)

(b) Inspection of graph of time plot of residual.
Cluster or uneven fluctuation (unfit).

Inspection of sample autocorrelation function of residuals.

4. $Y_t = -\alpha Y_{t-1} + \alpha^2 Y_{t-2} + Z_t$

(i) AR(2) process

$$(1 + \alpha B + \alpha^2 B^2) Y_t = Z_t$$

$$1 + \alpha\lambda - \alpha^2 \lambda^2 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2\alpha}$$

$\lambda > 1$

$$\alpha < \frac{\sqrt{5} - 1}{2}$$

(ii) $Y_t = -\alpha Y_{t-1} + \alpha^2 Y_{t-2} + Z_t$

$\gamma_0 \; cov(Y_t, Y_t) = -\alpha\gamma_1 + \alpha^2\gamma_2 + \sigma^2$ — ①

$\gamma_1 = cov(Y_t, Y_{t-1}) = -\alpha\gamma_0 + \alpha^2\gamma_1$ — ②

$\gamma_2 = cov(Y_t, Y_{t-2}) = -\alpha\gamma_1 + \alpha^2\gamma_0$ — ③

From ②

$$\gamma_1 = -\frac{\alpha\gamma_0}{1-\alpha^2} \quad — ④$$

Replace ④ into ③

$$\gamma_2 = -\alpha\left(-\frac{\alpha\gamma_0}{1-\alpha^2}\right) + \alpha^2\gamma_0$$

$$(2\alpha^2 - \alpha^4) \cdot \frac{\gamma_0}{1-\alpha^2} \quad — ⑤$$

$$\gamma_0 = \frac{\sigma^2(1-\alpha^2)}{1-2\alpha^2-2\alpha^4+\alpha^6}$$

$$\gamma_1 = \frac{\sigma^2(-\alpha)}{1-2\alpha^2-2\alpha^4+\alpha^6}$$

$$\gamma_2 = \frac{\sigma^2(2\alpha^2-\alpha^4)}{1-2\alpha^2-2\alpha^4+\alpha^6}$$

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5. (i) $(X_n - X_{n-1}) = \alpha(X_{n-1} - X_{n-2}) + \epsilon_n$

$$\alpha = \rho_1 = \frac{\sum_{3}^{200} (X_i - X_{i-1})(X_{i-1} - X_{i-2})}{\sum_{i=2}^{200} (X_i - X_{i-1})^2}$$

$$\alpha = \frac{587.83}{936.49} = 0.6277$$

$$\gamma_0 = \frac{1}{200} \sum_{i=2}^{200} (X_i - X_{i-1})^2$$

$$= \frac{936.49}{200} = 4.6825$$

$$\gamma_0 = \frac{\sigma^2}{1-\rho_1^2} = \frac{\sigma^2}{1-\alpha^2}$$

$$\hat{\sigma}^2 = 1 - \hat{\alpha}^2 \gamma_0$$

$$(1 - 0.6277^2)\, 4.6825 = 2.8376$$

(ii) X_{201}

$$(\hat{X}_{200}(1) - X_{200}) = \hat{\alpha}(X_{200} - X_{199})$$

$$= 0.6277(1.93 - 0.82) = 0.6967$$

$$\hat{X}_{200}(1) = X_{200} + 0.6967$$

$$= 1.93 + 0.6907 = 2.6267$$

6. (i) Forecasting future values of process
Description of the data
Construction of a model which fits the data
Deciding whether the process is out of control and required action.

(ii) AR(1) → Yes
MA(1) → No
AR(2) → No
Random walk → Yes

(iii) $X_t = a + bt + Y(t)$

$E(X_t) = a + bt + E(Y(t))$

Y_t is stationary
$E(Y_t) = \text{constant}$

a. $E(X_t) = a + bt + c$ which depends on t

so not stationary

b. $E(\nabla X_t) = E(X_t) - E(X(t-1))$

$a + bt + c - (a + b(t-1) + c)$
$= b$
Mean does not depends on t

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(iv) $X_1(t) - X_2(t)$

$= \alpha X_1(t-1) - X_2(t-1) + b\,(X_2(t-1) - X_1(t-1) +$

$e_1(t) - e_2(t))$

$= (a-b)\,[X_1(t-1) - X_2(t-1)) + (e_1(t) - e_2(t))$

$Y_t = X_1(t) - X_2(t)$

$Y_t = AR(1)$ process

7. i) Autoregressive process (AR)

~~It is~~ An auto regressive process with order p is a sequence of random variable (X_t) defined ~~consequen~~ consecutively by the rule.

$$X_t = \mu + \alpha_1(X_{t-1} - \mu) + \alpha_2(X_{t-2} - \mu) + \cdots \alpha_p(X_{t-p} - \mu) + \epsilon_t$$

Thus, the AR(p) model try to explain the current value of X as a linear combination of past values with some additional externally generated random variable.

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(b) Moving average process (MA)

A moving average process with order q is a sequence defined by

$$X_t = \mu + e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \dots \beta_q e_{t-q}$$

The MA(q) model explains the relations b/w X_t as an indirect effect, arising from the fact that the current value of the process results from recently past random errors.

(c) Autoregressive moving average process (ARMA)

The two basic processes (AR and MA) are combined together to give (ARMA)

$$X_t = \mu + \alpha_1 (X_{t-1} - \mu) + \alpha_2 (X_{t-2} - \mu) + \dots \alpha_p (X_{t-p} - \mu) +$$

$$e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \dots \beta_q e_{t-q}$$

(ii) a) MA(1) => stationary ARIMA(0, 0, 1)

(b) ARMA (2, 3)
Cannot be differenced
=> ARIMA (2, 0, 3)

$$(1 - 1.4B^2) X_t = e_t + 0.5 e_t$$

$$\lambda = \pm 0.8452$$

$$\lambda \leq 1$$

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(c) ARMA (2, 1)

$$X_t - 1.4X_{t-1} + 0.4X_{t-2} = e_t - e_{t-1}$$

$$(X_t - X_{t-1}) - 0.4(X_{t-1} - X_{t-2}) = e_t - e_{t-1}$$

$$\nabla X_t - 0.4\nabla X_{t-1} = e_t + e_{t-1}$$

ARIMA (1, 1, 1)

(ii) $\gamma_0 = cov(X_t, X_t)$

$= cov(e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3}, e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3})$

$= \sigma^2 + 0.25^2\sigma^2 + 0.5^2\sigma^2 + 0.25^2\sigma^2$

$= 1.375\sigma^2$

$\gamma_1 = cov(X_t, X_{t-1})$

$= cov(e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3}, e_{t-1} + 0.25e_{t-2} + 0.5e_{t-3} + 0.25e_{t-4})$

$= 0.25\sigma^2 + 0.5(0.25)\sigma^2 + 0.5(0.25)\sigma^2$

$= 0.5\sigma^2$

$\gamma_2 = cov(X_t, X_{t-2})$

$cov(e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3}, e_{t-2} + 0.25e_{t-3} +$

$0.5e_{t-4} + 0.25e_{t-5})$

$= 0.5\sigma^2 + 0.25^2\sigma^2 = 0.5625\sigma^2$

$\gamma_3 = cov(X_t, X_{t-3})$

$cov(e_t + 0.25e_{t-1} + 0.5e_{t-2} + 0.25e_{t-3}, e_{t-3} + 0.25e_{t-4} +$

$0.5e_{t-5} + 0.25e_{t-6})$

$= 0.25\sigma^2$

$\gamma_k = 0 \quad k > 3$

$\rho_0 = 1$

$\rho_1 = 0.364$

$\rho_2 = 0.404$

$\rho_3 = 0.183$

$\rho_k = 0 \quad k > 3$

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8. (i) $X_t = A_t + B_t$

$A_t = 0.5 A_{t-1} + 0.5 B_{t-1} + e_1^{(A)}$

$B_t = 0.7 A_{t-1} - 0.7 A_{t-2} + e_1^{(B)}$

$$Y_t = \begin{pmatrix} A_t \\ B_t \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.7 & 0 \end{pmatrix} \begin{pmatrix} A_{t-1} \\ B_{t-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -0.7 & 0 \end{pmatrix} \begin{pmatrix} A_{t-2} \\ B_{t-2} \end{pmatrix} + \begin{pmatrix} e_1^{(A)} \\ e_1^{(B)} \end{pmatrix}$$

$$Y_t = \begin{pmatrix} A_t \\ B_t \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.7 & 0 \end{pmatrix} Y_{t-1} + \begin{pmatrix} 0 & 0 \\ -0.7 & 0 \end{pmatrix} Y_{t-2} + \begin{pmatrix} e_1^{A} \\ e_1^{B} \end{pmatrix}$$

$$\lambda^2 - 0.5\lambda - 0.35 = 0$$

$$\lambda < 1$$

$Y_t \Rightarrow$ stationary

(ii) (a)

$$X_t = (\alpha + 1) X_{t-1} - (\alpha + 0.25\alpha^2) X_{t-2} + 0.25\alpha^2 X_{t-3} + e_t$$

$$X_t - X_{t-1} = \alpha (X_{t-1} - X_{t-2}) - 0.25\alpha^2 (X_{t-2} - X_{t-3}) + e_t$$

$$Y_t = \alpha Y_{t-1} - 0.25\alpha^2 Y_{t-2} + e_t$$

$X_t = \text{ARIMA}(2,1,0)$

$$\left(1-\alpha\beta+0.25\alpha^2\beta^2\right)Y_t = e_t$$

$$\beta = 2/\alpha$$

$\lambda > 1$ $\alpha < 2$

ARIMA (2,1,2)

(b) $Cov(Y_t, e_t) = cov(e_t, e_t) = \sigma^2$

$\gamma_0 = Cov(Y_t, Y_t)$

$\alpha\gamma_1 - 0.25\alpha^2\gamma_2 + \sigma^2$ — (1)

$\gamma_1 = \alpha\gamma_0 - 0.25\alpha^2\gamma_1$ — (2)

$\gamma_2 = \alpha\gamma_1 - 0.25\alpha^2\gamma_0$ — (3)

$\gamma_k = \alpha\gamma_{k-1} - 0.25\alpha^2\gamma_{k-2}$

from (2) $\gamma_1 = \dfrac{\alpha\gamma_0}{1+0.25\alpha^2}$ — (4)

Replace (3) into (1)

$\gamma_0 = \alpha\gamma_1 - 0.25\alpha^2(\alpha\gamma_1 - 0.25\alpha^2\gamma_0) + \sigma^2$

$1-(0.25\alpha^2)^2\gamma_0 = \dfrac{\alpha^2(1-0.25\alpha^2)}{1+0.25\alpha^2}\gamma_0 + \sigma^2$

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$$\gamma_0 = \frac{1 + 0.25\alpha^2}{(1 - 0.25\alpha^2)^3}\sigma^2$$

10. (i) $p = 1$ $q = 4$

ARMA (1, 4)

(ii) Non-linear non stationary time series model includes.

Bilinear models are those that exhibit bursty behavior

$$X_n - \alpha(X_{n-1} - \mu) = \mu + e_n + \beta e_{n-1} + b(X_{n-1} - \mu)e_{n-1}$$

Threshold autoregressive model are used to model cyclical ~~[illegible]~~ behaviors

(iii)

$$X_n = \mu + \alpha_1 (X_{n-1} - \mu) + e_n$$
$$X_n = \mu + \alpha_2 (X_{n-1} - \mu) + e_n$$

(iii) $X_t = e_t + \beta e_{t-1}$

$Var(X_t) = var(e_t + \beta e_{t-1})$

$var(e_t) + \beta^2 var(e_{t-1})$

$\sigma^2 + \beta^2 \sigma^2$

$(1+\beta^2)\sigma^2$

$\Delta Y_t = (0.6 + 0.3t + X_t) - (0.6 + 0.3(t-1) + X_{t-1})$

$= 0.3 + (X_t - X_{t-1})$

11. (i) Description of data
forecasting future ~~data~~ values
deciding whether the process is out of control
Construction of a model which fits data

(ii) Univariate
It is a sequence of obs (X_t) recorded at regular interval. The state space is continuous but time is discrete

Invertibility
A process is invertible if we can write white noise term as a convergent sum of X terms

Markov

A time series has markov property if :

$$P([X_t = a \mid X_1 = x_1, \ldots X_x = x_x]) = P(X_t = a \mid \cancel{X}_{\cancel{x}} = x)$$

(ii) 1. Seasoning differencing.
2. Method of moving average.

1. Least square trend removal
2. Differencing

12. (i) By presence of deterministic effect
(deterministic trend & seasonal effect)
If process is the integrated version of a more fundamental process.

(ii) a) ARIMA (p, d, q)

$$(1-B)^d \phi B (X_t - \mu) = \theta (B) e_t$$

(b) Tentative identification of a model from ARIMA
Estimation of the parameter
Diagnostic checks

(c) $d=1$

(d) (i) $X_t = 0.8e_{t-1} + e_t$

MA(1)
ARIMA (0, 0, 1)

(ii) $X_t = 2X_{t-1} + e_t + 0.5e_{t-4}$

(iii) $X_t = 1.5X_{t-1} + 0.2X_{t-2} + e_{t-1} + e_{t-2}$

ARMA (2, 1)

$X_t - 1.5X_{t-1} - 0.2X_{t-2} = e_t + e_{t-2}$.

13 (i) A weakly stationary process has constant mean and covariance is constant for each lag

(ii) $X_n = Z_n + \beta Z_{n-1}$

$E(X_n) = (1+\beta)\mu$

$Var(X_n) = Var(Z_n + \beta Z_{n-1})$

$= Var(1+\beta^2)\sigma^2$

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(iii) $X_n = 1.5X_{n-1} - 0.5X_{n-2} + Z_n$

$(1 - 1.5B + 5B^2)X = 2$

$(1-B)(1-0.5B)$

$1 = b \qquad B = 1/0.5$

not stationary

(iv) $X = ARIMA(1,1,0)$