Page No.
Date :

SRM-2
Assignment-2
Roll no : 408.

1)
(i) {Bid (B), Offer (O)}

$$\begin{array}{c} B \\ O \end{array} \begin{pmatrix} -\lambda & \lambda \\ \mu & \mu \end{pmatrix}$$

(ii) The holding times are exponentially distributed with parameter λ in state B, and μ in state O.

(iii) $\frac{\partial}{\partial t} {}_tP_s^{BB} = -\lambda \cdot {}_tP_s^{BB} + \mu_t P_s^{BO}$

$\frac{\partial}{\partial t} {}_tP_s^{BO} = \lambda_t P_s^{BB} - \mu_t P_s^{BO}$

We have two state model so.

${}_tP_s^{BB} + {}_tP_s^{BO} = 1$

$\therefore$

$\frac{\partial}{\partial t} {}_tP_s^{BB} = -\lambda_t P_s^{BB} + \mu.(1 - {}_tP_s^{BB});$

$\frac{\partial}{\partial t} [\exp((\lambda+\mu)t) . {}_tP_s^{BB}] = \mu \exp((\lambda+\mu)t)$

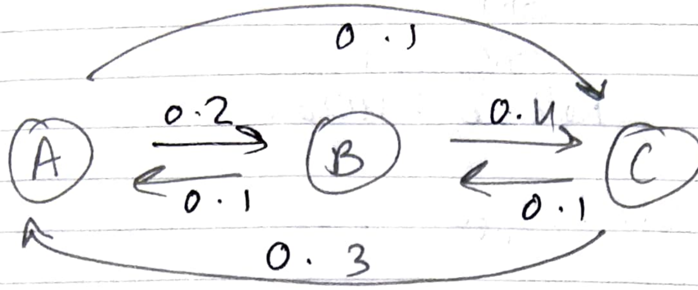
hence,

$\exp((\lambda+\mu)t) . {}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} \exp((\lambda+\mu)t) +$ constant

Since the process is in state Bid at time s (i.e $t=0$), the constant is $\frac{\lambda}{\mu+\lambda}$

$\therefore {}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \exp(-(\lambda+\mu)t)$

2) (i)



(ii) $\frac{d}{dt} P_{AA}(t) = -0.3 P_{AA}(t) + 0.1 P_{AB}(t) + 0.3 P_{AC}(t)$

$\frac{d}{dt} P_{AB}(t) = 0.2 P_{AA} t - 0.5 P_{AB}(t) + 0.1 P_{AC}(t)$

$\frac{d}{dt} P_{AC}(t) = 0.1 P_{AA}(t) + 0.4 P_{AB}(t) - 0.4 P_{AC}(t)$

(iii) Either, To stay in state A the eqn reduces to

$\frac{d}{dt} P_{\overline{AA}}(t) = -0.3 P_{\overline{AA}}(t)$

which has solution

$P_{\overline{AA}}(t) = \exp(-0.3t)$

So for $t=2$ we have $\exp(-0.6) = 0.5488$

OR

We can model this as poisson with parameter $(0.1+0.2) \times 2 = 0.6$

$P(Poi(0.6)=0) = \frac{e^{-0.6} 0.6^0}{0!}$

$= e^{-0.6} = 0.5488$

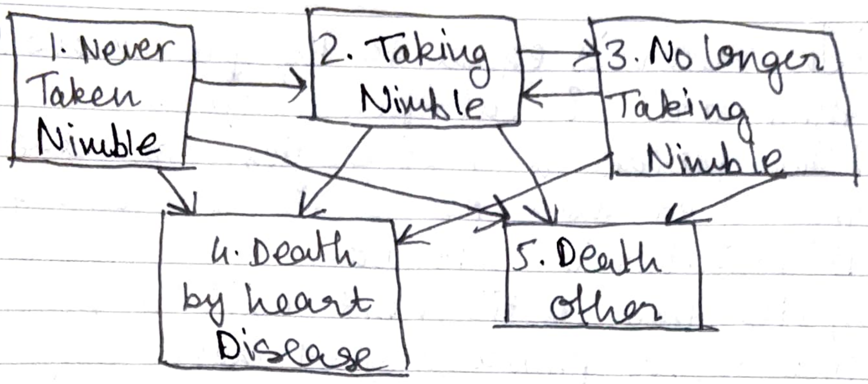
(iv) The only paths under which the third jump is into state C are BAC, CAC and CBC. The probability of each jump are given by the ratio of the transition rates. So the probabilities of each path are:

$BAC = \frac{2}{3} \cdot \frac{1}{5} \cdot \frac{1}{3} = \frac{2}{45}$ $\therefore$ Sum $= \frac{7}{36} = 0.194$

$CAC = \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{12}$

$CBC = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{4}{5} = \frac{1}{15}$

3) (i)



(ii) Using the Markov assumption OR the Chapman Kolmogorov equation is

$${}_{dt+t}P_x^{34} = {}_tP_x^{31}\,{}_{dt}P_{x+t}^{14} + {}_tP_x^{32}\,{}_{dt}P_{x+t}^{24} + {}_tP_x^{33}\,{}_{dt}P_{x+t}^{34} + {}_tP_x^{34}\,{}_{dt}P_{x+t}^{44} + {}_tP_x^{35}\,{}_{dt}P_{x+t}^{54}.$$

Since ${}_{dt}P_{x+t}^{54} = {}_tP_x^{31} = 0$

$${}_{dt+t}P_x^{34} = {}_tP_x^{32}\,{}_{dt}P_{x+t}^{24} + {}_tP_x^{33}\,{}_{dt}P_{x+t}^{34} + {}_tP_x^{34}\,{}_{dt}P_{x+t}^{44}$$

given that ${}_{dt}P_{x+t}^{44} = 1$, and assuming that for all small dt.

$${}_{dt}P_{x+t}^{ij} = \mu_{x+t}^{ij}\,dt + o(dt) \quad i \neq j$$

where $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$

then substituting we have

$${}_{dt+t}P_x^{34} = {}_tP_x^{32}\,\mu_{x+t}^{24}\,dt + {}_tP_x^{33}\,\mu_{x+t}^{34}\,dt + {}_tP_x^{34} + o(dt)$$

$$\therefore {}_{dt+t}P_x^{34} - {}_tP_x^{34} = {}_tP_x^{32}\,\mu_{x+t}^{24}\,dt + {}_tP_x^{33}\,\mu_{x+t}^{34}\,dt + o(dt).$$

$$\therefore \frac{d}{dt}\left({}_tP_x^{34}\right) = \lim_{dt \to 0} \frac{{}_{t+dt}P_x^{34} - {}_tP_x^{34}}{dt}$$

$$= {}_tP_x^{32}\,\mu_{x+t}^{24} + {}_tP_x^{33}\,\mu_{x+t}^{34}.$$

4) (i) The mean is equal to the parameter, so there are 3 calls per hour.

(ii) The process is memoryless so the fact that fred has not had a call for 15 minutes is irrelevant.
Expected time until next call is 20 minutes.

(iii) This is the probability of zero calls in time 0.5 hours. Using: $P_j(t) = e^{-\lambda t}(\lambda t)^j / j!$
OR

Since $P_0(0.5) = \frac{e^{-1.5}(1.5)^0}{0!}$,

$P_0(0.5) = e^{-1.5} = 0.2231$

(iv) The expected time that fred is on the phone call is the expected no. of calls times the expected lenght of a call.
Per hour this is 3 calls times 7 minutes
= 21 minutes
So the probability that the phone is engaged is $21/60 = 0.35$.

5) Using the Markov assumption or the Chapman Kolmogrov eqn is

$$P_{HH}(x, t+dt) = P_{HH}(x,t)\,P_{HH}(t, t+dt) + P_{HS}(x,t)\,P_{SH}(t, t+dt) + P_{HD}(x,t)\,P_{DH}(t, t+dt)$$

But $P_{DH}(t, t+dt) = 0$ or other explanation why path through D can be ignored.
So:

$$P_{HH}(x, t+dt) = P_{HH}(x,t)\,P_{HH}(t, t+dt) + P_{HS}(x,t)\,P_{SH}(t, t+dt)$$

Assuming that for all small dt.

$P_{ij}(t, t+dt) = \lambda_{ij}(t)dt + o(dt) \quad i \neq j$

$P_{ii}(t, t+dt) = 1 + \lambda_{ii}(t)dt + o(dt)$

Page No.
Date :

OR

$$P_{ii}(t, t+dt) = 1 - \sum_{j \neq i} \lambda_{ij}(t)dt + o(dt)$$

where the λs are the instantaneous transition rates and $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$

then substituting we have

$$P_{HH}(x, t+dt) = P_{HH}(x,t)(1 - \sigma(t)dt - \mu(t)dt) + P_{HS}(x,t)\rho(t, C_t) + o(dt)$$

so that

$$P_{HH}(x, t+dt) - P_{HH}(x,t) = P_{HH}(x,t)(-\sigma(t) - \mu(t))dt + P_{HS}(x,t)\rho(t, C_t)dt + o(dt)$$

and hence.

$$\frac{d}{dt} P_{HH}(x,t) = \lim_{dt \to 0} \frac{P_{HH}(x, t+dt) - P_{HH}(x,t)}{dt}$$

$$= P_{HH}(x,t)(-\sigma(t) - \mu(t)) + P_{HS}(x,t)\rho(t, C_t)$$

(ii) The equation simplifies when considering $P_{\overline{HH}}(t)$ to $\frac{d}{dt} P_{\overline{HH}}(0,t) = -(\sigma(t) + \mu(t))P_{\overline{HH}}(t)$

$$\frac{1}{P_{\overline{HH}}(0,t)} \frac{d}{dt} P_{\overline{HH}}(0,t) = -(\sigma(t) + \mu(t))$$

$$= \frac{d}{dt} \ln P_{\overline{HH}}(t)$$

Integrate both sides:

$$\left[\ln P_{\overline{HH}}(0,t)\right]_0^t = \int_{s=0}^{t} -(\sigma(s) + \mu(s))ds$$

as $P_{\overline{HH}}(0) = 1$

$$P_{\overline{HH}}(0,t) = \exp - \left(\int_{s=0}^{t} (\sigma(s) + \mu(s))ds\right)$$

Page No.
Date :

6) All three processes have a discrete state space. A Markov jump chain and Markov chain both operate in discrete time but a Markov jump process operates in continuous time. All have the Markov property which which is either that the future development of the process can be predicted from its present state alone, without reference to its past history. OR that.

$$P[X_t \in A \mid X_{s_1} = x_1, X_{s_2} = x_2, \ldots, X_{s_n} = x_n, S_s = x]$$
$$= P[X_t \in A \mid X_s = x]$$

for all times $s_1 < s_2 < \ldots < s_n < s < t$, all states $x_1, x_2, \ldots x_n, x$ in S and all subsets A of S. Either if a Markov Jump Process X is examined only at the times of its transitions, the resulting process is called Jump chain associated with X. OR for a jump process X the jump chain X shows the states visited by X, taking an identical path through the state space. The jump chain observes the Markov property and behaves as a Markov chain except when the jump chain encounters an absorbing state. From that time it makes no further transitions, implying that time stops for the Jump chain. The jump chain associated with X takes the same path through the state space as X does.

Page No.
Date :

1) (i) The MLE of the transition intensity from state i to state j is the no. of transitions from state i to state j divided by the total waiting time in state i. To estimate the transition intensities exactly we therefore need the total time spent in each state OR entry and exit times for each individual for each state, and the total no. of transitions of each type made.

(ii) $\frac{\partial}{\partial t} P_{AA}(s,t) = -P_{AA}(s,t)\,\mu$

$\frac{\partial}{\partial t} P_{AT}(s,t) = P_{AA}(s,t)\,\mu$

OR

$\frac{\partial}{\partial t} P(s,t) = P(s,t)\,M$

where, $M = \begin{pmatrix} -\mu & \mu \\ 0 & 0 \end{pmatrix}$ in order Active, theft.

OR

Integrated forward equations.

$P_{AA}(s,t) = \exp\left(-\int_{u=s}^{t} \mu\, du\right)$

$P_{AT}(s,t) = \int_{u=0}^{t} P_{AA}(s,u)\,\mu \cdot 1\, du$

(iii) Measure from time zero i.e $s=0$ and drop s from notation either.

$\frac{1}{P_{AA}(t)} \frac{\partial}{\partial t} P_{AA}(t) = -\mu$

$\frac{\partial}{\partial t} (\ln(P_{AA}(t))) = -\mu,$

hence. $P_{AA}(t) = \exp(-\mu t + c)$

As $P_{AA}(0) = 1$, $c = 0$, so

$P_{AA}(t) = \exp(-\mu t)$

A claim occurs with cost £c if moves to state "Theft claim."

Hence the expected ~~claim~~ cost is $C(1-\exp(-\mu T))$

OR

Solving for P_{AT}, we have

$$\frac{\partial}{\partial t} P_{AT}(t) = P_{AA}(t)\,\mu = (1-P_{AT}(t))\,\mu$$

(as the model has only 2 states)

Using an integrating factor, we can write

$$\frac{\partial}{\partial t}\left(\exp(\mu t) P_{AT}(t)\right) = \mu \exp(\mu t),$$

$$\exp(\mu t) P_{AT}(t) = \exp(\mu t) - 1,$$

$$P_{AT}(t) = 1 - \exp(-\mu t),$$

and hence the expected cost is $C(1-\exp(-\mu T))$

OR solving the integrated forward equation.

$$P_{AT}(T) = \int_{s=0}^{T} \exp(-\mu s)\,\mu\, ds$$

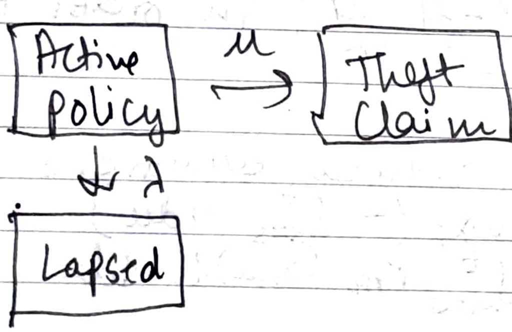
$$= \left[-\exp(\mu s)\right]_0^T$$

$$= 1 - \exp(-\mu T)$$

and hence, the expected cost is

$$C(1-\exp(-\mu T)).$$

(iv)



(v) $$\frac{\partial}{\partial t} P_{AA}(t) = -P_{AA}(t)(\mu+\lambda).$$

So, $P_{AA}(t) = \exp(-(\mu+\lambda)t)$.

We want. $$\frac{\partial}{\partial t} P_{AT}(t) = P_{AA}(t)\,\mu$$

$$= \mu \exp(-(\mu+\lambda)t).$$

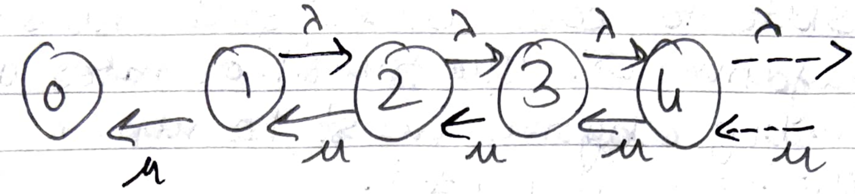
$$\therefore P_{AT}(t) = \left|\frac{-\mu}{(\mu+\lambda)} \exp(-(\mu+\lambda)t)\right|_0^T$$

$$= \frac{\mu}{\mu+\lambda}\left(1-\exp(-(\mu+\lambda)T)\right)$$

So claims become $\frac{\mu}{\mu+\lambda} C(1-\exp(-(\mu+\lambda)T))$.

8) (i) $\{0, 1, 2, 3, 4 \ldots\}$

(ii)



(iii) Generator Matrix.

$$
\begin{array}{c}
\text{Lives} \quad 0 \qquad 1 \qquad 2 \qquad 3 \qquad 4 \quad \ldots \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & \cdot \\
\mu & -(\mu+\lambda) & \lambda & 0 & 0 & \\
0 & \mu & -(\mu+\lambda) & \lambda & 0 & \\
0 & 0 & \mu & -(\mu+\lambda) & \lambda & \\
0 & 0 & 0 & \mu & -(\mu+\lambda) & \\
\cdot & \cdot & & & & \cdot
\end{bmatrix}
\end{array}
$$

(iv) Either; If a Markov Jump Process X_t is examined only at the times of transitions the resulting process is called the jump chain associated with X_t. OR

A jump chain is each distinct state visited in the order visited where the time set is the times when states are moved between.

(v)

$$
\begin{array}{c}
\text{Lives} \quad 0 \qquad 1 \qquad 2 \qquad 3 \qquad 4 \quad \ldots \\
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & \text{etc} \\
\mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 & 0 & \\
0 & \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 & \\
0 & 0 & \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & \\
0 & 0 & 0 & \mu/(\mu+\lambda) & 0 & \\
\text{etc} & & & & &
\end{bmatrix}
\end{array}
$$

(vi) $\left(\dfrac{\mu}{\mu+\lambda}\right)^3$

9) (i) The transition rates of moving from each state to each other state must be non-negative OR The transition rates off the leading diagonal must be non-negative OR $\mu_{ij} >= 0$ for $i \neq j$.

- The transition rates ON the leading diagonal must be non-positive OR $\mu_{ii} <= 0$.
- The sum of each row must be zero OR $\mu_{ii} = -\sum_{j \neq i} \mu_{ij}$
- Should be a finite or countable square Matrix.

(ii) State Space {2, 1, 0} policies in force

(iii)

$$2 \xrightarrow{0.4} 1 \xrightarrow{0.2} 0$$

(iv)
$$\frac{d}{dt} P_{22}(t) = -0.4 P_{22}(t)$$

$$\frac{d}{dt} P_{21}(t) = 0.4_{22}(t) - 0.2 P_{21}(t)$$

$$\frac{d}{dt} P_{20}(t) = 0.2 P_{21}(t)$$

10) (i)
$$\frac{d}{dt} P_{\overline{AA}}(t) = -2t \times P_{\overline{AA}}(t)$$

$$\frac{d}{dt}[\ln P_{\overline{AA}}(t)] = -2t$$

$$\ln P_{\overline{AA}}(s) = -s^2 + \text{constant}.$$

we know $P_{\overline{AA}}(0) = 1$, hence constant $= 0$

Hence, $P_{\overline{AA}}(s) = \exp^{-s^2}$

P(in first visit to B at time T | in state A at $t = 0$)

$$= \int_0^T P(\text{remains in A to time s}) \times P(\text{transition to B in time s, s + ds}) \times P(\text{remains in B to time T})\, ds$$

$$= \int_{s=0}^{T} P_{\bar{A}\bar{A}}(s) \times 2s \times P_{\bar{B}\bar{B}}(s,T)\, ds$$

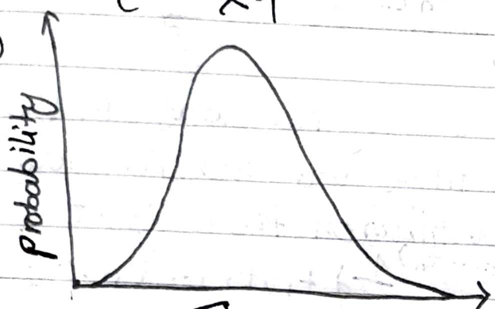
Using the result from part (i) and the similar result for P_{BB} with boundary condition $P_{BB}(s,s) = 1$, this gives us.

$$= \int_{s=0}^{T} e^{-s^2} \times 2s \times e^{-T^2+s^2}\, ds$$

$$= \int_{s=0}^{T} 2s \times e^{-T^2}\, ds$$

$$= e^{-T^2} \times T^2$$

(iii) (a)



(b) Initially probability increases from 0 at $T = 0$, and accelerates as the transition rate from A to B increases. However, as transition increase, it becomes more likely that the process has aldready visited state B and jumped backed to A. Therefore, the probability of being in the first visit to B tends (exponentially) to zero.

(c) Differentiate to find turning point:

$$\frac{d}{dt}\left[e^{-t^2} \times t^2\right] = 2t \times e^{-t^2} - 2t^3 \times e^{-t^2}$$

Set derivative equal to zero

$$e^{-t^2} \times 2t \times (1 - t^2) = 0$$

implies $t = 1$ for a positive solution. and, from above analysis, this is clearly a maximum.

Page No.
Date :

1) (i) let N_t denote the no. of claims up to time t.
Since, the poisson process has stationary increments we may take $t=0$, so that the reqd. conditional distribution is.

$$P(T_0 \le y \mid N_s = 1) = \frac{P(T_0 \le y, N_s = 1)}{P(N_s = 1)}$$

$$= \frac{P(N_y = 1, N_s - N_y = 0)}{P(N_s = 1)}$$

But $N_s - N_y$ is independent of N_y and has the same distribution as N_{s-y}.

$$\therefore \frac{(\lambda y e^{-\lambda y}) e^{-\lambda(s-y)}}{\lambda s e^{-\lambda s}} = \frac{y}{s}$$

which is the cdf of the uniform Distn on $[0, s]$.

(ii) Since holding times are independent, each having an exponential distribution their joint density is

$$\lambda^n e^{-\lambda(t_1+t_2+\dots t_n)} \mathbb{1}_{\{t_1, t_2, \dots t_n > 0\}}$$

(iii) $$P(N_s = k \mid N_t = n) = \frac{P(N_s = k, N_t = n)}{P(N_t = n)}$$

$$= \frac{P(N_s = k, N_t - N_s = n-k)}{P(N_t = n)}$$

Using again that the poisson process has stationary and independent increments, and the no. of claims in an interval $[0, t]$ is Poisson (λ, t) we derive from above that

$$P[N_s = k \mid N_t = n) = \frac{\frac{e^{-\lambda s}(\lambda s)^k}{k!} \cdot \frac{e^{-\lambda(t-s)} \lambda^{n-k} (t-s)^{n-k}}{(n-k)!}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{e^{-\lambda t} \lambda^n s^k (t-s)^{n-k}}{k!(n-k)!} \cdot \frac{n!}{e^{-\lambda t} \lambda^n t^n}$$

$$= \frac{n!}{k!(n-k)!} \cdot \frac{s^k (t-s)^{n-k}}{t^k t^{n-k}}$$

$$= \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}$$

which is binomial with parameters n and s/t.