Probability & Statistics Assignment - II.
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Q1. Let X be claim size of home insurance

Let Y be claim size of car insurance

$X \sim N(800, 100^2)$
$Y \sim N(1200, 300^2$

$P(Y_1 + Y_2 + Y_3 > X_1 + X_2 + X_3 + X_4 + 800)$

$P(Y_1 + Y_2 + Y_3 - (X_1 + X_2 + X_3 + X_4) > 800)$

$\sim N(3 \times 1200 - 4 \times 800), 300^2(3) + 100^2$

$\sim N(400, 310000)$

$P\left(Z > \frac{800 - 400}{\sqrt{310000}}\right)$

$P(Z > 0.71842) =$
$1 - P(Z < 0.71842)$
$1 - P(Z < 0.72)$
$1 - 0.76424$
$= 0.23576$

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Q.2.) $N \sim$ Neg. binomial (k, p).

$$P(N=n) = {}^{n+2}C_n (0.9)^3 (0.1)^n$$

$$= {}^{(n+3)-1}C_{n-1} (0.9)^3 (0.1)^n$$

here, $k=3$, $p=0.9$

$X \sim$ Gamma $(6, 2)$

$$M_y(t) = M_N(\log M_x(t))$$

$y \rightarrow$ Total claim amount.

(i) MGF of y

$$M_y(t) = E(e^{yt})$$
$$= E(E(e^{(x_1+x_2 \cdots x_N)t}/N))$$

$$E(e^{(x_1+x_2+\cdots x_N)t}/N=n) = E(e^{tx_1}) \cdot E(e^{tx_2}) \cdots E(e^{tx_N})$$

$$= \left(1-\frac{t}{2}\right)^{-6n}$$

$$E(E(e^{yt}/N=n)) = E\left[\left(1-\frac{t}{2}\right)^{-6n}\right]$$

$$= E\left(e^{n\log\left(1-\frac{t}{2}\right)^{-6}}\right)$$

$$= M_N\left[\log\left(1-\frac{t}{2}\right)^{-6}\right].$$

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$$M_N(t) = \left(\frac{pe^t}{1-qe^t}\right)^k$$

$$= \left[\frac{p\, e^{\log(1-\frac{t}{2})^{-6}}}{1-q\,e^{\log(1-\frac{t}{2})^{-6}}}\right]^k$$

$$\left[\frac{p\left(1-\frac{t}{2}\right)^{-6}}{1-q\left(1-\frac{t}{2}\right)^{-6}}\right]^3 = \left[\frac{0.9\left(1-\frac{t}{2}\right)^{-6}}{1-0.1\left(1-\frac{t}{2}\right)^{-6}}\right]^3$$

(ii) $V(y) = E(N)\, V(X) + [E(X)]^2\, V(N)$

$$= \frac{3}{0.9} \times \frac{6}{4} + \left(\frac{6}{2}\right)^2 \times \frac{3(0.1)}{(0.9)^2}$$

Standard deviation $= 2.881$.

P.3- $f(x) = \lambda e^{-\lambda(x-\alpha)}$, $x \geq \alpha$ and $\lambda, \alpha > 0$

MGF $= E[e^{xt}] \rightarrow \int_\alpha^\infty e^{tx}\,\lambda e^{-\lambda(x-\alpha)}\,dx$

$$M_X(t) = \lambda \int_\alpha^\infty e^{tx}\, e^{-\lambda x}\, e^{\lambda\alpha}\, dx$$

$$= e^{\lambda\alpha}\lambda \int_\alpha^\infty e^{-(\lambda - t)x}\, dx$$

$$= \frac{e^{\alpha\lambda}\lambda}{\quad} \left[e^{-(\lambda-t)x}\right]_\alpha^\infty$$

$$= \frac{e^{\lambda\alpha}\lambda}{t-\lambda}\left[-e^{-(\lambda-t)\alpha}\right]$$

$$= e^{t\alpha}\frac{\lambda}{\lambda-t}$$

(ii) $M_x(t) = \frac{\lambda e^{t\alpha}}{\lambda-t} = \lambda e^{t\alpha}(\lambda-t)^{-1}$

$$M'_x(t) = \lambda\left(e^{t\alpha}(\lambda-t)^{-2} + (\lambda-t)^{-1}e^{t\alpha}(\alpha)\right)$$

$$= \lambda(\cancel{\alpha}(\lambda-t)^{-1}e^{t\alpha}(\alpha) + e^{t\alpha}(\lambda-t)^{-2})$$

$$= \lambda e^{t\alpha}(\lambda-t)^{-2}\left[\alpha(\lambda-t)+1\right]$$

$$= \lambda\lambda^{-2}\left[\alpha\lambda+1\right]$$

$$= \frac{1}{\lambda}\left[\alpha\lambda+1\right] = \frac{\lambda\alpha+1}{\lambda}$$

$$M''_x(t) = \lambda\left[\alpha e^{t\alpha}(\lambda-t)^{-2} + \alpha^2(\lambda-t)^{-1}e^{t\alpha} + e^{t\alpha}(+2)(\lambda-t)^{-3} + \alpha(\lambda-t)^{-2}e^{t\alpha}\right]$$

put $t=0$

$$\lambda\left[\alpha(\lambda)^{-2} + \alpha^2(\lambda)^{-1} + 2(\lambda)^{-3} + \alpha(\lambda)^{-2}\right]$$

$$E(x^2) = \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2}$$

$$V(X) = E(X^2) - \left[E(X)\right]^2$$

$$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left[\alpha + \frac{1}{\lambda}\right]$$

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Q.4.

$$G_V(t) = \left[\frac{p}{(1-qt)}\right]^k$$

$$G_V(t) = \Sigma t^v P(V=v)$$

$$P(V=4) = \frac{\sqrt{k+4}}{\sqrt{5}\,\sqrt{k}}\, p^k q^k$$

$$\rightarrow \frac{(k+3)!}{(k-1)!\,4!}\, p^k (1-p)^k$$

Q.5. $f(x,y) = a\,x(x+y)$

$$P\left(y > \frac{1}{3}x + 10\right)$$

$$a\int_{10}^{20}\int_{0}^{5} (x^2 + xy)\,dx\,dy = 1$$

$$\int_{10}^{20} \left(\frac{x^3}{3} + \frac{x^2}{2}\,y\right)_0^5 dy = \frac{1}{a}$$

$$\int_{10}^{20} \frac{125}{3} + \frac{25}{2}\,y\,dy = \frac{1}{a}$$

$$= \left(\frac{125}{3}y + \frac{25}{4}y^2\right)_{10}^{20} = \frac{1}{a}$$

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$$a\left[\frac{2500}{3}+2500-\frac{1250}{2}-625\right]$$

$$a\left[\frac{1250}{3}+1875\right]=1$$

$$a\left(\frac{6875}{3}\right)=1$$

$$a=\frac{3}{6875}$$

$$\therefore a=0.000436364.$$

(iii) $f_{y/x}=\frac{f(x,y)}{f_x(x)}$

$$f_x(x)=\frac{3}{6875}\int_{10}^{20}x^2+xy\,dy=$$

$$=\frac{3}{6875}\left[x^2y+\frac{xy^2}{2}\right]_{10}^{20}$$

$$=\frac{3}{6875}\left[20x^2+200x-10x^2-50x\right]$$

$$=\frac{3}{6875}\left[10x^2+150x\right]=\frac{30x}{6875}(x+15)$$

$$f_{x,y}(x,y)=\frac{3x(x+y)\times 6875(x+15)}{6875(30)x}$$

$$=\frac{x+y}{10(x+15)}$$

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$$E(y/x) = \int_{10}^{20} y \cdot \frac{(x+y)}{10(x+15)}\, dy$$

$$= \frac{1}{10(x+15)} \int_{10}^{20} xy + y^2\, dy$$

$$= \frac{1}{10(x+15)} \left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_{10}^{20}$$

$$= \frac{1}{10(x+15)} \left[200x + \frac{8000}{3} - 50x - \frac{1000}{3} \right]$$

$$= \frac{1}{10(x+15)} \left[150x + \frac{7000}{3} \right]$$

$$\frac{1}{x+15} \left[15x + \frac{700}{3} \right]$$

(ii) $P\left(y > \frac{1}{3}, x+10\right)$

$$f(y) = \frac{3}{6875} \int_0^5 x^2 + xy\, dx$$

$$= \frac{3}{6875} \left[\frac{x^3}{3} + \frac{x^2}{2} y \right]_0^5$$

$$= \frac{3}{6875} \left(\frac{125}{2} + \frac{25}{2} y \right)$$

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$$\frac{3}{\cancel{56}6875}\int_{\frac{x}{3}+10}^{20} \frac{125}{3}+\frac{25}{2}y \, dy$$

$$=\frac{3}{6875}\left[\frac{125y}{3}+\frac{25y^2}{4}\right]_{\frac{1}{3}x+10}^{20}$$

$$=\frac{3}{6875}\left[\frac{125}{3}(20)+\frac{25}{4}(400)-\frac{125}{3}\left(\frac{1}{3}x+10\right)-\frac{25}{4}\left(\frac{1}{3}x+10\right)^2\right]$$

$$\frac{3}{6875}\left[\frac{2500}{3}+2500-\frac{125x}{9}-\frac{1250}{3}-\frac{25}{4}\left(\frac{1}{9}x^2+100+\frac{2x}{3}\right)\right]$$

Q.6. $X \sim Poi(\lambda)$ $Y \sim Poi(\mu)$

$M_X(t) = e^{\lambda(e^t-1)}$ $M_Y(t) = e^{\mu(e^t-1)}$

$Z = X-Y$

$$M_Z(t) = E\left[e^{tZ}\right] = E\left[e^{t(X-Y)}\right]$$
$$= E\left(e^{tX} \cdot e^{-tY}\right)$$

$$= \frac{E\left(e^{tX}\right)}{E\left(e^{tY}\right)} = \frac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}}$$

$$= e^{(\lambda-\mu)(e^t-1)}$$

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So, By uniqueness MGF of $Z = e^{(\lambda-\mu)(e^t-1)}$

$\sim Poi(\lambda-\mu)$

Let $Z = X+Y$

$E(e^{tZ}) = e^{\lambda(e^t-1)} \times e^{\mu(e^t-1)}$

$= e^{(\lambda+\mu)(e^t-1)}$

By uniqueness $Z \sim Poi(\lambda+\mu)$

Q.7.

(i) $V(X/Y=2) = E(X^2/Y=2) - [E(X-Y)]^2$

$E(X^2/Y=2) = \sum x^2 \cdot \frac{P(X=x, Y=2)}{P(Y=2)}$

~~When $x=1$, when $x=2$, when ..., when $x=4$~~

~~$(1)^2 \frac{P(X=1, Y=2)}{P(Y=2)}$ $(2)^2 \frac{P(X=2, Y=2)}{P(Y=2)}$~~

When $x=1$

$1^2 \frac{P(X=1, Y=2)}{P(Y=2)} + (2)^2 \frac{P(X=2, Y=2)}{P(Y=2)} + (3)^2 \frac{P(X=3, Y=2)}{P(X=2)}$

$+ (4)^2 \frac{P(X=4, Y=2)}{P\,Y=2}$

$= 8.6231$

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$E(X/y=2)$

$$= \frac{\sum x\, P(X=x, y=y)}{P(y=y)}$$

$$= \frac{(0)}{0.6}$$

$$(1)\frac{P(X=1, y=2)}{P(y=2)} + (2)\frac{(P(X=1)), y=2)}{P(y=2)} + 3\frac{(P(X=3, y=2)}{P(y=2)}$$

$$+ 2\frac{P(X=4, y=2)}{P(y=2)}$$

$$= \frac{0.6 + 0.2 + 0.8}{0.6} = 2.8333,$$

$\therefore V(X/y=2) = 0.8015471$

(ii) $f_{y}(u, v) = \frac{48}{67}(2uv - u^2)$

$$E(u/v=v) = \frac{f(u,v)}{f(v)}$$

$$f(v) = \frac{48}{67}\int_0^1 2uv - v^2\, du = \frac{48}{67}\left[u^2v^2 - \frac{u^3}{3}\right]_0^1$$

$$= \frac{48}{67}\left[v - \frac{1}{3}\right] = \frac{16}{67}(3v-1)$$

$$f(v) = \frac{16}{67}(3v-1)$$

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$$\frac{f(u,v)}{f(v)} = \frac{40}{67} \frac{(2uv^2 - u^2)}{\times 16(3v-1)} \times 67$$

$$= \frac{3}{3v-1} (2uv^2 - u^2) . \quad 2u^2v^2 - u^3$$

$$E(u/v=v) = \frac{3}{3v-1} \int_0^1 u(12uv^2 - u^2)\,du$$

$$= \frac{3}{3v-1} \left[\frac{2u^3v^2}{3} - \frac{v^4}{4} \right]_0^1 = \frac{3}{3v-1} \left[\frac{2}{3}v^2 - \frac{1}{4} \right]$$

$$= \frac{3}{3v-1} \left(\frac{8v^2-3}{12} \right) = \frac{8v^2-3}{4(3v-1)}$$

8.8. ~~gamma~~ $X \sim \text{gamma}(3, 2)$

$$E(X) = \frac{3}{2}, \quad V(X) = \frac{3}{4}$$

$$E(Y) = E(E(Y/X=x)) = E(3x+1)$$

$$= 3(E(X)) + (1) = 3 \times \frac{3}{2} + 1$$

$$= \frac{11}{2}$$

$$V(Y) = E(V(X/Y)) + V(E(X/Y))$$

$$= E(2x^2 + 5) + V(3x+1)$$

$$= 2E(X)^2 + 5 + 9V(X)$$

$$= V(X) = E(X)^2 - (E(X))^2$$

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$[E(X)^2] = \frac{3}{4} + \frac{9}{4}$

$= \frac{12}{4} = 3$

$V(Y) = 2(3) + 5 + \ldots + 7 \times \frac{3}{4}$

$= 11 + \frac{27}{4} = \frac{44+27}{4}$

$= \frac{71}{4}$

Q.9. $E(X) = 3$, $V(X) = 4$
$E(Y) = 4$, $V(Y) = 1$

correlation $= -0.3\sqrt{4} = -0.6 = Cov(X,Y)$

(i) $Cov(X, X+Y) = Cov(X,X) + Cov(X,Y)$

$= V(X) + Cov(X,Y)$

$= 4 - 0.6$
$= 3.4$

(ii) $var(Z) = Cov(X+Y, X+Y)$

$= Cov(X,X) + 2Cov(X,Y) + Cov(Y,Y)$

$= V(X) + 2(Cov(X,Y)) + V(Y)$

$= 4 + 2(-0.6)) + 1$

$= 3.8$

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Q.10. $f_{x,y}(x,y) = \cancel{\frac{k}{x} e^{-\alpha} \, y/\beta} \quad k x^{-\alpha} e^{y/\beta}$

$$k \int_{1}^{\infty} \int_{1}^{\infty} x^{-\alpha} e^{-y/\beta} \, dx \, dy = k$$

$$k \int e^{-y/\beta} \left[\frac{x^{-(\alpha-1)}}{-\alpha+1} \right]_{1}^{\infty} = 1$$

$$\frac{k}{1-\alpha} \int e^{-y/\beta} \, [-1]$$

$$= \frac{k}{\alpha-1} \int e^{-y/\beta} \, dy = 1$$

$$= \frac{k}{\alpha-1} \frac{\left[e^{-y/\beta}\right]_{1}^{\infty}}{\left(\frac{-1}{\beta}\right)} = \frac{k}{\alpha-1} (\beta) \, (+e^{-1/\beta}) = 1$$

$$k = \frac{\alpha-1}{\beta\left(e^{-1/\beta}\right)}$$