

Q₁

- i) A Stochastic Model has the capacity to handle uncertainties in the inputs applied. Stochastic model possess some ~~set of~~ inherent randomness - the same set of Parameter value and initial conditions will lead to an ensemble of different outputs.

Adv of Stochastic models:

- Stochastic model can reflect real-world economic scenarios that provide range of possible outcomes you may experience and the relative likelihood of each.
- By running thousands of calculations using many different estimates of future economic conditions, stochastic models predict a range of possible future investment results showing the potential upside and downside of each.

- ii) The time spent in state H before the next visit to S has mean σ^{-1} . Therefore for a reasonable estimate for σ is the reciprocal of the mean length of each visit = (Number of transition from H to S), although it would be equally valid to use MLE, which is similarly for P .

Q2.
1) Consider a Markov chain taking values in the set
 $S = \{i = 0, 1, 2, 3, 4\}$; where i represents the number of umbrellas in the place where the Actuary currently is (at home or office).

if $i = 1$ and it rains he takes the umbrella, move to the other place, where there are already 3 umbrellas and including the one he brings, therefore he will now have 4 umbrellas.

Thus, $P_{1,4} = P$, because P is the probability of rain.

if $i = 1$ but does not rain then he do not take the umbrella, goes to the other place and find 3 umbrellas.

Thus, $P_{1,3} = 1 - P = q$.

However, if $i = 0$, he must move to other place where 4 umbrellas are kept with probability $P_{0,4} = 1$

similarly for the other states, The process is depicted by the following dig.



ii)

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q & P \\ 0 & 0 & q & P & 0 \\ 0 & q & P & 0 & 0 \\ q & P & 0 & 0 & 0 \end{vmatrix}$$

d iii) For stationary distribution $\pi = \pi P$.

$$\pi(0) = \pi(4) \cdot q \quad - (1)$$

$$\pi(1) = \pi(0)q + \pi(4)p \quad - (2)$$

$$\pi(2) = \pi(2)q + \pi(3)p \quad - (3)$$

$$\pi(3) = \pi(1)q + \pi(2)p \quad - (4)$$

$$\pi(4) = \pi(0) + \pi(1)p \quad - (5)$$

Substituting 1 in 5.

$$\begin{aligned} \pi(4) &= \pi(4)q + \pi(1)p \Rightarrow \pi(1) = \pi(4) \cdot (1-q)/p \\ \Rightarrow \pi(1) &= \pi(4) \cdot p/q \Rightarrow \pi(0) = \pi(4). \end{aligned}$$

Sub 6 in 2.

$$\pi(1) = \pi(3) \cdot q + \pi(4)p \Rightarrow \pi(4) = \pi(3)q + \pi(4) \cdot p.$$

$$\pi(3) = \pi(4) \cdot (1-p)/q \Rightarrow \pi(3) = \pi(4) \quad - (7)$$

Sub 7 in 3.

$$\pi(2) = \pi(2) \cdot q + \pi(3) \cdot p \Rightarrow \pi(2) = \pi(3) \cdot p/(1-q).$$

$$\pi(2) = \pi(3) \cdot p/(1-q) \Rightarrow \pi(2) = \pi(3) \quad - (8)$$

Substituting 6 & 7 in 4.

$$\pi(3) = \pi(1)q + \pi(2)p \Rightarrow \pi(4) = \pi(1)q + \pi(4)p$$

$$\Rightarrow \pi(1) = \pi(4)(1-p)/q \Rightarrow \pi(1) = \pi(4)$$

Therefore $\pi(1) = \pi(3) = \pi(4) = \pi(0)/q$.

$$\text{But } \pi(0) + \pi(1) + \pi(2) + \pi(3) + \pi(4) = \pi(0)/q.$$

$$\text{So } \pi(4) \cdot q + 4\pi(4) = 1.$$

$$\text{Therefore } \pi(4) = 1/(4+q) = \pi(1) = \pi(2) = \pi(3).$$

$$\text{and } \pi(0) = q/(4+q)$$

(10) He will be wet every time he happens to be in state 0 and it rains. The chance he is in state 0 is $\pi(0)$. The chance it rains is p . hence probability that he gets wet.

$P(\text{wet}) = \text{Probability of being in state 0 and it raining}$

$$P(\text{wet}) = (q/(q+p))P.$$

v.) $P(\text{wet}) = 0.6 \times 0.4 / (0.4 + 0.6) = 0.4 \text{ s\%}$

if he ~~has~~ want the chance to be ~~at~~ less than 1%. then, clearly.

he need more umbrellas. he, suppose he need N umbrellas. set up the markov chain as above and generalising. it is clear that

$$\pi(1) = \pi(2) = \pi(3) = \pi(4) : \dots = \pi(N) = \pi(0)/q.$$

Put $\pi(0) + \pi(1) + \pi(2) + \pi(3) + \pi(4) + \dots + \pi(N) = 1$.

Substituting in terms of $\pi(N)$.

$$\Rightarrow \pi(N) \cdot q + N\pi(N) = 1.$$

$$\pi(N) = \dots$$

Therefore probability of getting wet $P(\text{wet}) = p \times \pi(0) =$

For $P(\text{wet}) < 1/100 \Rightarrow pq / (N+q) < 1/100$.

$N > 100 \times p \times q - p$; given $p = 0.6, q = 0.4$.

~~25~~
~~1) : ii~~

Q3.

i) ii).

Process	State Space	Time domain
Counting Process	Discrete	Discrete or Continuous
General Random walk	Discrete or Continuous	Discrete
Poisson process	Discrete	Continuous
Markov chain	Discrete	Discrete
Markov Jump	Discrete	Continuous

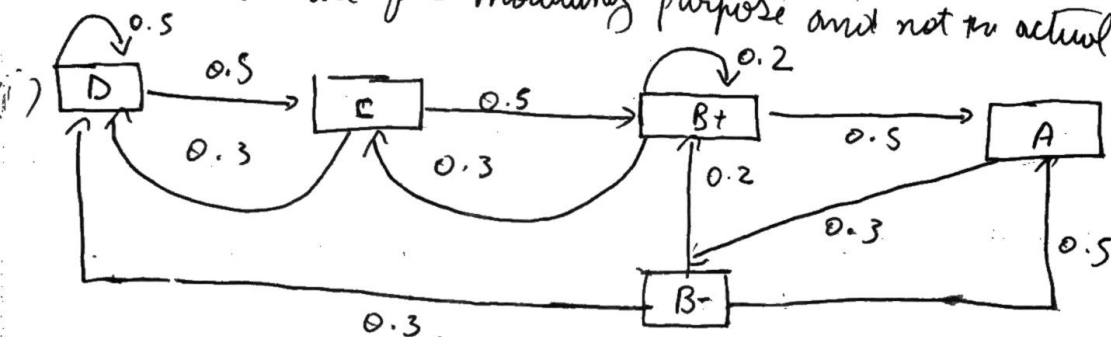
ii).

- a) Number of times the account has been overdrawn since it was opened - Poisson Process / Counting Process.
- b) Status (overdrawn or in credit) of the account on the last day of each month - Markov Jump Chain / Markov Process.
- c) Number of direct debits paid since the account was opened. Poisson.
- d) Status (overdrawn or in credit) of the account at any time since the account was opened. Markov Jump Process

29.)

i) Post history is needed to decide where to go in the chain. If a sportsman is at A and his/her performance reduces, you need to know what level of performance he was at the previous year to determine where he or she drops one or two levels.

ii) The B level needs to be split into two. B+ is the level with no reduction in performance parameter last year. B- is the level with reduction in performance parameter last year. These levels are for modelling purpose and not the actual levels.



iv) The Transition matrix is:

	D	C	B+	B-	A
D	0.5	0.5	0	0	0
C	0.3	0.2	0.5	0	0
B+	0	0.3	0.2	0	0.5
B-	0.3	0	0.2	0	0.5
A	0	0	0	0.3	0.7

V) stationary distribution

The stationary distribution is the set of probabilities.

that satisfy the matrix equation $\pi = \pi P$

with an additional condition $\sum \pi v = 1$

written out fully, this set of matrix equation is

$$0.5 \pi \pi_{DD} + 0.3 \pi \pi_{CC} + 0.3 \pi \pi_{BB} = \pi \pi_{DD} \quad \text{--- (a)}$$

$$0.5 \pi \pi_{DD} + 0.2 \pi \pi_{CC} + 0.3 \pi \pi_{BB} = \pi \pi_{CC} \quad \text{--- (b)}$$

$$0.5 \pi \pi_{CC} + 0.2 \pi \pi_{BB} + 0.2 \pi \pi_{BB} = \pi \pi_{BB} \quad \text{--- (c)}$$

$$0.3 \pi \pi_{AA} = \pi \pi_{BB} \quad \text{--- (d)}$$

$$0.5 \pi \pi_{BB} + 0.5 \times 0.3 \pi \pi_{AA} + 0.7 \pi \pi_{AA} = \pi \pi_{AA} \quad \text{--- (e)}$$

sub. (d) in (e).

$$0.5 \pi \pi_{BB} + 0.5 \times 0.3 \pi \pi_{AA} + 0.7 \pi \pi_{AA} = \pi \pi_{AA}$$

$$\Rightarrow \pi \pi_{BB} = 0.3 \pi \pi_{AA} \quad \text{--- (vi)}$$

sub. $\pi \pi_{BB}$; $\pi \pi_{CC}$ aaaaaa $\pi \pi_{AA}$ in (c) we get

$$0.5 \pi \pi_{CC} + 0.2 \times 0.3 \pi \pi_{AA} + 0.2 \times 0.3 \pi \pi_{AA} = 0.3 \pi \pi_{AA}$$

$$\Rightarrow \pi \pi_{CC} = 0.36 \pi \pi_{AA}$$

substituting $\pi \pi_{BB}$, $\pi \pi_{CC}$ aaaaaa $\pi \pi_{AA}$ in (b)

$$0.5 \pi \pi_{DD} + 0.2 \times 0.36 \pi \pi_{AA} + 0.3 \times 0.3 \pi \pi_{AA} = 0.3 \pi \pi_{AA}$$

$$\Rightarrow \pi \pi_{CC} = 0.36 \pi \pi_{AA}$$

substituting $\pi \pi_{BB}$, $\pi \pi_{CC}$ all in $\sum \pi v = 1$

$$\pi \pi_{AA} + 0.5 \pi \pi_{AA} + 0.3 \pi \pi_{AA} = 0.36 \pi \pi_{AA} \quad \text{--- (vii)}$$

$$\pi \pi_{AA} = 0.42445$$

$$\pi \pi_{BB-} = 0.12733$$

$$\pi \pi_{BB+} = 0.12733$$

$$\pi \pi_{CC} = 0.15280$$

$$\pi \pi_{DD} = 0.16808$$

(VI) Long run average interest value is ~~100% π_0 + 120% π_C~~

$$100\% \pi_0 + 120\% \pi_C + 150\% (\pi_{B+} + \pi_{B-}) + 175\% \pi_A$$

$$= 1.4762 \text{ million USD}$$

VII) Let m_i be the number of transitions (and years) taken to reach level A from any current level i . Then

$$m_{DD} = 1 + 0.5 m_{DD} + 0.5 m_{CC} \quad \text{--- (a)}$$

$$m_{CC} = 1 + 0.2 m_{CC} + 0.5 m_{BB+} + 0.3 m_{DD} \quad \text{--- (b)}$$

$$m_{BB+} = 1 + 0.2 m_{BB+} + 0.5 m_{AA} + 0.3 m_{CC} \quad \text{--- (c)}$$

$$m_{BB-} = 1 + 0.3 m_{DD} + 0.5 m_{AA} + 0.2 m_{BB+} \quad \text{--- (d)}$$

$$m_{AA} = 0$$

from (a) we get $m_{DD} = 2 + m_{CC}$.

Substituting in (b) we get.

$$m_{CC} = 1 + 0.2 m_{CC} + 0.5 m_{BB+} + 0.3 (2 + m_{CC})$$

$$m_{BB+} = m_{CC} - 3.2$$

Substituting in (c)

$$m_{BB+} = m_{CC} - 3.2$$

Substituting in (d)

$$m_{BB+} = 1 + 0.2 m_{BB+} + 0.5 m_{AA} + 0.3 m_{CC}$$

$$m_{CC} - 3.2 = 1 + 0.2 (m_{CC} - 3.2) + 0.3 m_{CC}$$

$$m_{CC} = \underline{7.12 \text{ years}} \quad \therefore m_{DD} = 2 + m_{CC} = \underline{9.12 \text{ years}}$$

Q5.

i) There is an explicit dependence on the past behaviour of $\{Y_j : j \leq n\}$ in the probability distribution of Y_{n+1} , further, X_n is nothing but the sum of Y_j . Hence the Markov property does not hold.

ii) In part i) we show that there is explicit dependence on the past behaviour of $\{Y_j : j \leq n\}$ and hence the Markov property does not hold, this implies that the sequence does not form a Markov chain.

iii) The transition matrix is given by

$$\begin{pmatrix}
 p & 1-p & 0 & \dots & \dots & 0 \\
 0 & pe^{-\lambda} & 1-pe^{-\lambda} & 0 & \dots & \dots \\
 \vdots & 0 & pe^{-2\lambda} & 1-pe^{-2\lambda} & 0 & \dots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 0 & \vdots & \vdots & \vdots & \vdots & \vdots
 \end{pmatrix}$$

iv)

- a) The chain is time homogeneous since the transition probabilities calculated in part i) is independent of time n .
- b) it is irreducible, since the number of errors can never go down.

© There are no recurrent states; hence there can be no stationary distribution.

Alternatively if a stationary distribution π exists, it has the following

$$\pi_0 p = \pi_0.$$

$$\pi_0 (1-p) + \pi_1 p e^{-\lambda} = \pi_1.$$

$$\pi_1 (1-p e^{-\lambda}) + \pi_2 p e^{-2\lambda} = \pi_2.$$

and so on.

Since $p \leq 1$ we have $\pi_0 = 0$ and then $\pi_1 = 0$, etc, hence no stationary distribution exists.

V) Probability of no further error is

$$p e^{-\lambda n} = p_n e^{-n\lambda}.$$

Q7

(a) The Process may be expressed as a Markov chain by considering following states. Each state indicates the

current number of successive defeats.

State number of successive defeats.

10 - Not defeated last time.

21 - Defeated in the last match.

32 - Defeated in the last two matches.

43 - Defeated in the last three matches.

54 - Captain sacked.

0.7	0.3	0	0	0
0.7	0	0.3	0	0
0.7	0	0	0.3	0
0.7	0	0	0	0.3
0	0	0	0	1

⑦

- (b) A Markov chain is said to be irreducible if any state j can be reached from any state i . The above process is not irreducible as the captain once sacked, can never become the captain again.

- (c) (i) The probability of remaining the captain for exactly four matches is given by.
- | Match # | #1 | #2 | #3 | #4 | Probability |
|-------------|------|------|------|------|------------------|
| Result | Lose | Lose | Lose | Lose | |
| Probability | 0.3 | 0.3 | 0.3 | 0.3 | $0.3^4 = 0.0081$ |

- (ii) The Probability of remaining the captain for exactly five matches is given by.

Match #	#1	#2	#3	#4	#5	Prob
Result	Not Lose	Lose	Lose	Lose	Lose	
Probability	0.7	0.3	0.3	0.3	0.3	0.00567

(iii) The Probability of remaining the captain for exactly seven matches is given by.

Match #	#1	#2	#3	#4	#5	#6	#7
Result	Any	Any	Not Lose	Lose	Lose	Lose	Lose
Probability	1	1	0.7	0.3	0.3	0.3	0.3

$\Rightarrow$ Probability 0.00567.

iv) The probability of remaining the captain for exactly nine matches is given by.

Match #	#1	#2	#3	#4	#5	#6	#7	#8	#9
Result	N.L	all four loses	N.L	L	L	L	L	L	L
Probability	$1 - 0.3^4 = 0.9919$		0.7	0.3	0.3	0.3	0.3	0.3	0.3

Probability = 0.0056241

(d) define N = Expected Number of matches as a captain given that the current state is i . since the captain is newly appointed,

the variable N in the question is N_1 as defined here. We have

$$N_1 = 1 + 0.7 + 0.3 \times N_2$$

$$N_2 = 1 + 0.7 \times N_1 + 0.3 \times N_3$$

$$N_3 = 1 + 0.7 \times N_1 + 0.3 \times N_4$$

$$N_4 = 1 + 0.7 \times N_1$$

Solving the above equations, we get

$$N_1 = 123.46$$

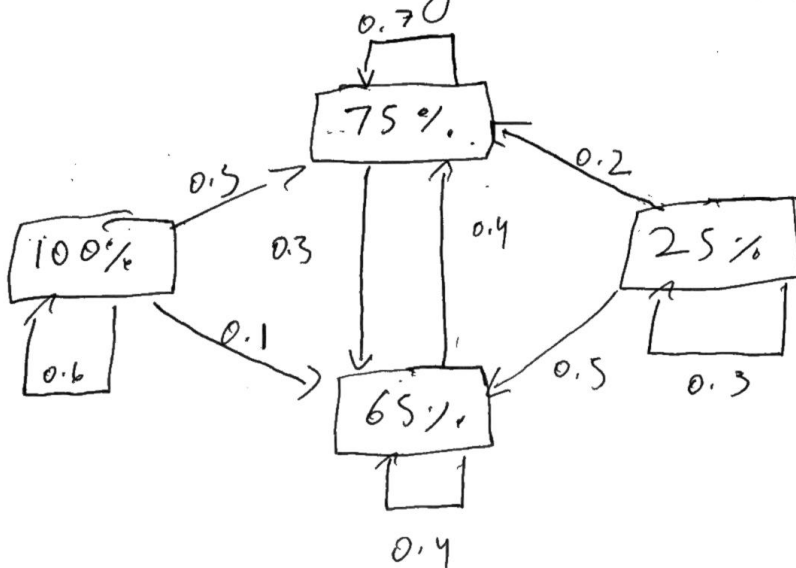
$$N_2 = 160.49$$

$$N_1 = 174.94$$

Therefore, $E(N)$ is 174.94 matches.

d7

Ⓐ The state transition diagram is set out as below.



Ⓑ We are to calculate the % of corporate buyer having a target % for XYZ of 65% - in 2 years time.

over the long run

The state of system after one year $S_1 = S_1 P$.

$$(0.05; 0.3, 0.45, 0.2) \begin{bmatrix} 0.6 & 0.3 & 0.1 & - \\ - & 0.7 & 0.3 & - \\ - & 0.4 & 0.4 & 0.2 \\ - & 0.2 & 0.5 & 0.3 \end{bmatrix} = (0.03, 0.445, 0.375, 0.15)$$

Hence the state of the system in 2 years time $S_2 = S_1 P$

$$(0.03, 0.445, 0.375, 0.15) \begin{bmatrix} 0.6 & 0.3 & 0.1 & - \\ - & 0.7 & 0.3 & - \\ - & 0.4 & 0.4 & 0.2 \\ - & 0.2 & 0.5 & 0.3 \end{bmatrix} \Rightarrow (0.018, 0.5005, 0.3615, 0.12)$$

Hence the % of corporate buyer having a target % of 65% for XYZ in 2 year time is 36.15%.

The long run steady state is found by solving the following equation

$$S = SP$$

$$(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4) P$$

$$(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4) \begin{pmatrix} 0.6 & 0.3 & 0.1 & - \\ - & 0.7 & 0.3 & - \\ - & 0.4 & 0.4 & 0.2 \\ - & 0.2 & 0.5 & 0.3 \end{pmatrix} \quad (5)$$

Hence we take the five equations.

$$x_1 = 0.6 x_1$$

$$x_2 = 0.3 x_1 + 0.7 x_2 + 0.4 x_3 + 0.2 x_4$$

$$x_3 = 0.1 x_1 + 0.3 x_2 + 0.4 x_3 + 0.5 x_4$$

$$x_4 = 0.2 x_3 + 0.3 x_4$$

$$x_1 + x_2 + x_3 + x_4 = 1$$

from the first eqn above.

$$0.4 x_1 = 0$$

$$\text{So, } x_1 = 0$$

sub into other eqn.

$$0.3 x_2 = 0.4 x_2 + 0.2 x_4 \quad (1)$$

$$0.6 x_3 = 0.3 x_2 + 0.3 x_4 \quad (2)$$

$$0.7 x_4 = 0.2 x_3 \quad - (3)$$

$$x_2 + x_3 + x_4 = 1 \quad (4)$$

one of the above eqn is redundant. Hence we have

$$x_1 = 0, \quad x_2 = 0.5423, \quad x_3 = 0.3559, \quad x_4 = 0.1017$$

c) The steady state proportion of customers having a 65% target allocation for XYZ Ltd is 55.59%. It is noted that the initially this proportion is 45% and it quickly drops to.

- 37.5% in one year time and
- 36.2% in two year time

It is ; therefore, likely that the steady state shall be reached in a few years time.

Q11) i) If each room is represented by the state, then the transition matrix P for this Markov chain is follows.

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

ii) The chain is irreducible because it is possible to go from any state to any other state. However, it is not a periodic, because for any even n , $P_{6,1}^n$ will be zero and for any odd $P_{6,5}^n$ will also be zero. This means that there is no power of P that would have all its entries strictly positive.

- (iii) For P to be stationary $\pi P = \pi$.
 perform matrix multiplication and show πP is equal to π .
- (iv) We find from π that the mean recurrence time, (i.e. the expected time to return), for the random walk is $1/\pi(1) = 12$.

v) Let $\psi(i) = E(\text{Number of steps to reach state } 5 | x_0 = i)$

We have $\psi(5) = 0$

$$\begin{aligned}\psi(6) &= 1 + (1/2)\psi(5) + (1/2)\psi(4) \\ &= 1 + (1/2)\psi(6) + (1/2)\psi(3) \\ \psi(3) &= 1 + (1/4)\psi(1) + (1/4)\psi(2) + (1/4)\psi(4) + (1/4)\psi(5) \\ &= 1 + \psi(3) + \psi(2)\end{aligned}$$

We solve and find $\psi(1) = 7$.

~~to find~~