Assignment - 2

Q1. X = claim size on a loan insurance
$X \sim Normal(800, 100^2)$
Y = claim size on a car insurance
$Y \sim Normal(1200, 300^2)$

$$P(Y_1+Y_2+Y_3 > X_1+X_2+X_3+X_4+800)$$
$$= P((Y_1+Y_2+Y_3)-(X_1+X_2+X_3+X_4) > 800)$$

$$(Y_1+Y_2+Y_3)-(X_1+X_2+X_3+X_4) \sim N[3(1200)-4(800), 3(300)^2+4(100)^2]$$
$$\sim N(400, 310000)$$

$$P((Y_1+Y_2+Y_3)-(X_1+X_2+X_3+X_4) > 800)$$

$$P\left(Z > \frac{800-400}{\sqrt{310000}}\right) = P(Z > 0.718)$$
$$= 1-P(Z < 0.718)$$
$$\simeq 1-0.76424$$
$$\simeq 0.23576$$

Q2. N = number of claims
$N \sim$ Negative Binomial $(3, 0.9)$

X = claim Amount
$X \sim Gamma(6, 2)$

Y = Total claim Amt.

PAGE NO. :
DATE : / /

$$M_Y(t) = M_N(\log M_X(t))$$

$$M_X(t) = \left(1 - \frac{t}{2}\right)^{-6}$$

$$\Rightarrow M_Y(t) = M_N\left(\log\left(1 - t/2\right)^{-6}\right)$$

$$\because M_N(t) = \left(\frac{pe^t}{1-qe^t}\right)^t$$

$$\therefore M_Y(t) = \left(\frac{pe^{\log\left(1-\frac{t}{2}\right)^{-6}}}{1-qe^{\log(1-t/2)^{-6}}}\right)^3$$

$$M_Y(t) = \left[\frac{p\left(1-\frac{t}{2}\right)^{-6}}{1-q\left(1-\frac{t}{2}\right)^{-6}}\right]^3$$

$$M_Y(t) = \left(\frac{0.9\left(1-t/2\right)^{-6}}{1-0.1\left(1-t/2\right)^{-6}}\right)^3$$

$$E(N) = \frac{3}{0.9} = \frac{10}{3}$$

$$V(N) = \frac{k(0.1)}{(0.9)^2} = \frac{3}{0.9}\left(\frac{1}{9}\right) = \frac{10}{27}$$

$$E(X) = \frac{\alpha}{\lambda} = \frac{6}{2} = 3 \quad \Bigg| \quad V(X) = \frac{\alpha}{\lambda^2} = \frac{6}{4} = \frac{3}{2}$$

$$V(Y) = E(N)\,V(X) + E(X)^2\,V(N)$$

$$= \frac{10}{\cancel{3}}\left(\frac{\cancel{3}}{2}\right) + (3)^2\left(\frac{10}{27}\right)$$

$$= 5 + \frac{10}{3} = 25/3$$

$$S.D.(Y) = \sqrt{\frac{25}{3}} = \underline{2.886}\ldots$$

Q3 $f_X(x) = \lambda e^{-\lambda(x-\alpha)}$

(i) $M_X(t) = E(e^{tx})$

$$= \int_{\alpha}^{\infty} e^{tx}\,\lambda e^{-\lambda(x-\alpha)}\,dx$$

$$= \lambda \int_{\alpha}^{\infty} e^{tx}\,e^{-\lambda x}\,e^{\lambda\alpha}\,dx$$

$$= \lambda e^{\lambda\alpha} \int_{\alpha}^{\infty} e^{x(t-\lambda)}\,dx$$

$$= \frac{\lambda e^{\lambda\alpha}}{t-\lambda}\left[e^{(t-\lambda)x}\right]_{0}^{\infty}$$

$$= \frac{\lambda e^{\lambda\alpha}}{t-\lambda}\left[e^{-(\lambda-t)x}\right]_{0}^{\infty}$$

$$= \frac{\lambda \cancel{e^{\lambda\alpha}}}{t-\lambda}\left(0 - \cancel{e^{-\alpha\lambda}}\cdot e^{t\alpha}\right)$$

$$= \frac{-\lambda e^{t\alpha}}{t-\lambda} = \frac{\lambda e^{t\alpha}}{\lambda - t}$$

PAGE NO. :

DATE : / /

(iii) $M'_X(t) = \lambda(e^{t\alpha})(\lambda-t)^{-2}(-1)(-1) + \lambda(\lambda-t)^{-1}(\alpha)(e^{t\alpha})$

$$= \frac{\lambda e^{t\alpha}}{(\lambda-t)^2} + \frac{\lambda\alpha e^{t\alpha}}{\lambda-t}$$

$$E(X) = M'_X(0) = \frac{\lambda(1)}{\lambda^2} + \frac{\lambda\alpha}{\lambda}$$

$$= \frac{1+\lambda\alpha}{\lambda}$$

$M''_X(t) = \lambda e^{t\alpha}(-2)(\lambda-t)^{-3}(-1) + \lambda\alpha(\lambda-t)^{-2}(e^{t\alpha})$
$+ \lambda\alpha e^{t\alpha}(\lambda-t)^{-2}(-1)(-1) + \lambda\alpha(\lambda-t)^{-1}(e^{t\alpha})(\alpha)$

$$= \frac{2\lambda e^{t\alpha}}{(\lambda-t)^3} + \frac{\lambda\alpha e^{t\alpha}}{(\lambda-t)^2} + \frac{\lambda\alpha e^{t\alpha}}{(\lambda-t)^2} + \frac{\alpha^2\lambda e^{t\alpha}}{\lambda-t}$$

$$= \frac{2\lambda e^{t\alpha}}{(\lambda-t)^3} + \frac{2\lambda\alpha e^{t\alpha}}{(\lambda-t)^2} + \frac{\lambda\alpha^2 e^{t\alpha}}{\lambda-t}$$

$$E(X^2) = M''_X(0) = \frac{2\lambda(1)}{\lambda^3} + \frac{2\lambda\alpha}{\lambda^2} + \frac{\lambda\alpha^2}{\lambda}$$

$$= \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2$$

$$V(X) = E(X^2) - (E(X))^2$$

$$= \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2 - \left[\frac{1}{\lambda^2} + \alpha^2 + \frac{2\alpha}{\lambda}\right]$$

$$= \frac{1}{\lambda^2}$$

DATE :

Q4. $G_v(t) = \left(\frac{p}{1-qt}\right)^k$, $p+q=1$

$\therefore$ given distribution is negative binomial type

$$P(V=4) = \frac{\sqrt{k+4}}{\sqrt{5}\sqrt{k}} p^4 q^4$$

Q5. $f(x,y) = ax^2 + axy$ $\quad 0 < x < 5$, $\quad 10 < y < 20$

$$\Rightarrow 1 = \int_{10}^{20} \int_0^5 ax^2 + axy \, dx \, dy$$

$$1 = \int_{10}^{20} \left\{ \left[\frac{ax^3}{3}\right]_0^5 + \left[ay\frac{x^2}{2}\right]_0^5 \right\} dy$$

$$1 = \int_{10}^{20} \left(\frac{125a}{3} + \frac{25ay}{2}\right) dy$$

$$1 = \left[\frac{125ay}{3} + \frac{25ay^2}{4}\right]_{10}^{20}$$

$$1 = \frac{1250a}{3} + 1875a$$

$$\boxed{a = \frac{3}{6875}}$$

DATE: / /

$$f_X(x) = \int_{10}^{20} (x^2+xy)\, a\, dy$$

$$= \frac{3}{6875}\int_{10}^{20} (x^2+xy)\,dy$$

$$= \frac{3}{6875}\left[x^2 y + \frac{xy^2}{2}\right]_{10}^{20}$$

$$= \frac{3}{6875}\left[10x^2 + 150x\right]$$

$$= \frac{6x(x+15)}{1375}$$

$$E(Y|X) = y\, f(Y|X) = y \int_{10}^{20} \frac{f(x,y)}{f_X(x)}$$

$$= \int_{10}^{20} \frac{y\,(\cancel{3/6875})(x^2+xy)\, dy}{(\cancel{3/6875})(10x^2+150x)}$$

$$= \cancel{\int_{10}^{20}} = \frac{x}{(10x^2+150x)}\int_{10}^{20} y(x+y)\,dy$$

$$= \frac{1}{10(x+15)}\left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_{10}^{20}$$

$$= \frac{1}{10(x+15)}\left[150x + \frac{7000}{3}\right]$$

$$= \frac{45x+700}{3(x+15)}$$

$$P\left(Y > \frac{1}{3}X + 10\right) \Rightarrow$$

$$f_Y(y) = \int_0^5 \frac{3}{6875}(x^2 + xy)\,dx$$

$$= \frac{3}{6875}\left(\frac{x^3}{3} + \frac{x^2 y}{2}\right)_0^5$$

$$= \frac{3}{6875}\left[\frac{125}{3} + \frac{25y}{2}\right]$$

$$= \frac{1}{6875}\left(\frac{250 + 75y}{2}\right)$$

$$P\left(Y > \frac{1}{3}X + 10\right) = \int_{\frac{1}{3}X+10}^{20} \frac{1}{6875} \times \frac{(250 + 75y)}{2}\,dy$$

$$= \frac{1}{2(6875)}\left[250y + \frac{75y^2}{2}\right]_{\frac{1}{3}X+10}^{20}$$

$$= \frac{1}{2(6875)}\left[250\left(20 - \frac{X}{3}\right) + \frac{75}{2}\left((20)^2 - \left(\frac{1}{3}X+10\right)^2\right)\right]$$

$$\cancel{= \frac{1}{2(6875)}\left[5000 - \frac{250X}{3} + 15K - \frac{75X^2}{9} - 75K - \frac{15KX}{3}\right]}$$

$$= \frac{1}{2(6875)}\left[2500 - \frac{250X}{3} + 15000 - \frac{25X^2}{3} - 7500 - \frac{15000X}{3}\right]$$

$$= \frac{1}{2(6875)}\left[10000 - \frac{1750}{3}X - \frac{25}{3}X^2\right]$$

Q6. $X \sim Poi(\lambda)$, $Y \sim Poi(\mu)$

$M_X(t) = e^{\lambda(e^t-1)}$ $\qquad$ $M_Y(t) = e^{\mu(e^t-1)}$

$$\Rightarrow M_{X-Y}(t) = E(e^{t(X-Y)})$$
$$= E(e^{Xt - Yt})$$
$$= \frac{E(e^{xt})}{E(e^{Yt})} = \frac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}}$$
$$= e^{(\lambda-\mu)(e^t-1)}$$

$\therefore X - Y \sim Poi(\lambda - \mu)$

→ Hence, proved

$$\Rightarrow M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{(tX+Yt)}) = E(e^{tX}) \cdot E(e^{Yt})$$
$$= e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)} = e^{(\lambda+\mu)(e^t-1)}$$

$\therefore X + Y \sim Poi(\lambda + \mu)$

→ Hence, proved

Q7. (i) $Var(X|Y=2) = E(X^2|Y=2) - (E(X|Y=2))^2$

$$= \sum_x x^2 \frac{P(X=x, Y=2)}{P(Y=2)} - \left(\sum_x x \frac{P(X=x, Y=2)}{P(Y=2)}\right)^2$$

$$= \frac{1^2(0) + 2^2(0.3) + 3^2(0.1) + 4^2(0.2)}{0.6} - \left[\frac{1(0) + 2(0.3) + 3(0.1) + 4(0.2)}{0.6}\right]^2$$

$$= \frac{53}{6} - \frac{289}{36} = \frac{29}{36}$$

PAGE NO. :
DATE : / /

Q8. $X \sim$ gamma $(3,2)$

$\rightarrow E(Y|X=x) = 3x+1$

$E(E(Y|X=x)) = E(Y)$

$E(Y) = E(3X+1)$

$= 3E(X)+1$

$= 3(3/2)+1$

$= 11/2$

$= \underline{\underline{5.5}}$

$\rightarrow Var(Y|X=x) = 2x^2+5$

$Var(Var(Y|X=x)) = V(Y)$

$V(Y) = E(Var(Y|X=x))$
$+ V(E(Y|X=x))$

$= E(2x^2+5) + V(3x+1)$

$= 2E(X^2) + 5 + 9V(X)$

$= 2(3) + 5 + \left(\frac{9 \times 3}{4}\right)$

$= 6+5 + \frac{27}{4} = \frac{71}{4}$

$= \underline{\underline{17.75}}$

Q9. $Z = X+Y$

$E(X) = 3$ $\qquad$ $E(Y) = 4$

$\sqrt{V(x)} = 2$ $\qquad$ $\sqrt{V(Y)} = 1$

(i.) $Cov(X,Z) = Cov(X, X+Y)$ $\qquad$ $cor(X,Y) = -0.3$

$= Cov(X,X) + Cov(X,Y)$

$= V(X) + (-0.6)$

$= 4 - 0.6$

$= \underline{\underline{3.4}}$

(ii.) $Var(Z) = Var(X+Y)$

$= V(X) + V(X) + 2Cov(X,Y)$

$= 4+1+2(-0.6)$

$= 5 + (-1.2)$

$= \underline{\underline{3.8}}$

PAGE NO.:
DATE: / /

Q10: $f(x,y) = Kx^{-\alpha} e^{-y/\beta}$, $1 < x < \infty$
$1 < y < \infty$
$\alpha > 1$, $\beta > 0$

$$1 = K \int_1^\infty x^{-\alpha}\,dx \int_1^\infty e^{-y/\beta}\,dy$$

$$1 = k \left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_1^\infty \left[\frac{e^{-y/\beta}}{-1/\beta}\right]_1^\infty$$

$$1 = K \left[0 - \frac{1}{-\alpha+1}\right] \left[0 + \beta\left(e^{(-1/\beta)}\right)\right]$$

$$1 = k \left(\frac{1}{-\alpha+1}\right) \left[-\beta\left(e^{(-1/\beta)}\right)\right]$$

$$K = \frac{-\alpha+1}{-\beta e^{(-1/\beta)}} = \frac{\alpha-1}{\beta e^{(-1/\beta)}}$$