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Assignment 1 [P & S]

Q1] $\therefore \hat{\lambda} = \frac{\sum x_i}{n}$

$\hat{\lambda} = 200$

$\therefore$ 95% CI of $\hat{\lambda}$

$= \hat{\lambda} \pm z_{2.5\%} \sqrt{\frac{\hat{\lambda}}{n}}$

$\therefore = 200 \pm 1.96 \times \sqrt{200}$

$\boxed{[172.2814, 227.718]}$

Q2] i) $E[\bar{x}] = E\left[\frac{\sum x}{n}\right]$

$= \frac{1}{n} \sum E[x]$

$= \frac{1}{\cancel{n}} \cdot \cancel{n}\mu = \mu$

$V[\bar{x}] = V\left[\frac{\sum x}{n}\right]$

$= \frac{1}{n^2} \sum V[x]$

$= \frac{1}{n^{\cancel{2}}} \cdot \cancel{n} \cdot \sigma^2$

$= \sigma^2/n$

ii) $E(s^2) = E\left(\frac{1}{n-1} \sum (x-\bar{x})^2\right)$

$E[x^2] = V[x] + E[x]^2$

$= \frac{1}{n-1} E\left(\sum x^2 - n\bar{x}^2\right)$

$= \frac{1}{n-1} E\left(\sum(\mu^2+\sigma^2) - n\left(\mu^2 + \frac{\sigma^2}{n}\right)\right)$

$= \frac{1}{n-1} E\left(\cancel{n\mu^2} + n\sigma^2 - \cancel{n\mu^2} - \sigma^2\right)$

$= \frac{1}{\cancel{n-1}} E\left(\cancel{(n-1)}\sigma^2\right) = \sigma^2$

iii) $\therefore \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$\therefore V[S^2] = 2(n-1)$

$V\left[\frac{(n-1)S^2}{\sigma^2}\right] = 2(n-1)$

$\frac{(n-1)^{\cancel{2}}}{\sigma^4} V[S^2] = 2(n-\cancel{1})$

$V[S^2] = \frac{2\sigma^4}{n-1}$

iv) $P[0.5\sigma^2 < S^2 < 0.5\sigma^2]$ $\quad E[S^2] = \sigma^2$

$\therefore P[0.5\times 100 < S^2 < \overset{1.5}{\cancel{0.5}}\times 100]$

$P[50 < S^2 < \overset{150}{\cancel{50}}]$

$P\left[\frac{50\times(9)}{100} < \chi^2_{9} < \frac{\overset{3}{\cancel{150}}\times 9}{2\cancel{100}}\right]$

$P[\chi^2_9 < \overset{13.5}{\cancel{4.5}}] - P[\chi^2_9 < 4.5]$

~~0.1245~~ $0.8587 - 0.1245$

$= 0.7342$

Q3] i) $L = [P(X=0)]^{86} \cdot [P(X=1)]^{75} \cdot [P(X=2)]^{16} \cdot [P(X=3)]^{2} \cdot [P(X=4)]^{1}$

$$\therefore = \left[\overset{^4C_0}{\cancel{\text{constant}}} \cdot p^0 \cdot (1-p)^4\right]^{86} \times \left[{}^4C_1 \times p \times (1-p)^3\right]^{75}$$

$$\times \left[{}^4C_2 \cdot p^2 \cdot (1-p)^2\right]^{16} \cdot \left[{}^4C_3 \cdot p^3 \cdot (1-p)\right]^2 \cdot$$

$$\left[{}^4C_4 \cdot p^4 \cdot (1-p)^0\right]$$

$$= \text{constant} \times p^{117} \times (1-p)^{603}$$

$$\log L = 117 \log p + 603 \log(1-p) + \text{constant}$$

$$\frac{d \ln L}{dp} = \frac{117}{p} + \frac{603}{(1-p)} = 0$$

$$\hat{p} = \frac{117}{720} = 0.1625$$

ii) H_0: ~~model data~~ claims are from binomial
H_1: are not from binomial

	0	1	2	3	4
O_i	86	75	16	2	1
E_i	88.542	68.724	19.998	2.574	0.108

$$T.S = \frac{\sum (O_i - E_i)^2}{E_i} \sim \chi^2_{5-1-1-2}$$

$$\therefore 1.24322 \sim \chi^2_1$$

$\therefore$ At 5% LOS we have insufficient evidence to reject H_0.

Q4] $f(x) = \frac{c\beta^3}{(x+\beta)^4}$; $x > 0$, $\beta > 0$

i] Comparing to PDF to Pareto Dist.

$$PDF = \frac{\alpha \lambda^\alpha}{(\lambda + x)^{\alpha+1}}$$

$\therefore$ β corresponds to λ

$C = 3$

$$E[X] = \frac{\lambda}{\alpha - 1} = \frac{\beta}{2} \qquad V[X] = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} = \frac{3\beta^2}{4}$$

ii] $\therefore E[X] = \bar{X}$

$$\frac{\beta}{2} = \bar{X}$$

$$\hat{\beta} = 2\bar{X}$$

$$\text{Bias} = E[\hat{\beta}] - \beta$$

$$= 2E[\bar{X}] - \beta$$

$$= \frac{2}{n}\sum E[X_i] - \beta$$

$$= \frac{\cancel{2}}{\cancel{n}} \cdot \cancel{n} \cdot \frac{\beta}{\cancel{2}} - \beta = 0$$

unbiased estimator

$$MSE = var(\hat{\beta}) + Bias^2$$

$$= V[2\bar{X}] = \frac{4\sum V[X]}{n^2}$$

$$= \frac{\cancel{4} \cdot \cancel{n}}{n^2} \cdot \frac{3\beta^2}{\cancel{4}}$$

$$= \frac{3\beta^2}{n}$$

$\therefore$ as $n \to \infty$

$MSE \to 0$

It is consistent.

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iii) $\hat{\beta} = b\bar{X}$

$E[\hat{\beta}] = b\,E[\bar{X}]$

$= \frac{b}{\alpha} \cdot \alpha \cdot \frac{\beta}{2}$

$= \frac{b\beta}{2}$

$V[\hat{\beta}] = b^2 V[\bar{X}]$

$= \frac{b^2}{n^2} \cdot \alpha \cdot \frac{3\beta^2}{4}$

$= \frac{3}{4} \frac{b^2\beta^2}{n}$

$MSE = V(\hat{\beta}) + Bias^2$

$= \frac{3}{4}\frac{b^2\beta^2}{n} + \frac{(b^2\beta^2 + 4\beta^2 - 4\beta^2 b)}{4}$

$M = \frac{\beta^2}{4n}(3b^2 + nb^2 + 4n - 4nb)$

$\frac{dM}{db} = \frac{\beta^2}{4n}(6b + 2nb - 4n) = 0$

$6b + 2nb - 4n = 0$

$\therefore b = \frac{2n}{3+n} = 2/(3+n/n)$

iv) $E(\hat{\beta}) = \frac{b\beta}{2}$

$= \frac{2}{(1+3/n)} \times \frac{\beta}{2}$

$= \frac{n\beta}{n+3}$

$Bias = E(\hat{\beta}_2) - \beta$

$= \frac{n\beta}{n+3} - \beta$

$= -\frac{3\beta}{n+3}$

The estimator underestimates the parameter.

$$MSE = \frac{\beta^2}{4n}(3b^2 + nb^2 + 4n - 4nb)$$

$$= \beta^2 / 4n \left[b^2(n+3) + 4n(1-b) \right]$$

$$= \frac{\beta^2}{4n}\left[\frac{4n^2}{n+3} + \frac{4n(3-n)}{n+3}\right]$$

$$MSE = \frac{12n\beta^2}{n+3}$$

$n \to \infty$. MSE doesn't tend to 0 $\therefore$ its inconsistent

Q5] i) $X \sim U(a, b)$

$$E[X] = \frac{a+b}{2} \qquad V[X] = \frac{1}{12}(b-a)^2$$

$$\bar{x} = \frac{a+b}{2} \qquad = S^2 = \frac{1}{12}(b-a)^2$$

$$2\bar{x} - b = a \rightarrow (1) \qquad 12s^2 = (b-a)^2 \rightarrow (2)$$

$$\sqrt{12s^2} = 2\bar{x} - 2a$$

$$\therefore \hat{a} = \bar{x} - \sqrt{3}s$$

ii) $b = 2\bar{x} - a$

$$12s^2 = (b-a)^2$$

$$2\sqrt{3}s = (b - 2\bar{x} + b)$$

$$\cancel{2}\sqrt{3}s = \cancel{2}(b - \bar{x})$$

$$\hat{b} = \bar{x} + \sqrt{3}s$$

iii) Sample observation

1, 3, 5, 7, 9

$\bar{x} = 5$

$S = 3.16227$

-0.4772

$\therefore \hat{a} =$ [illegible]

$\hat{b} = 10.477$

$\therefore \hat{b}$ is better estimator th[an]

$\hat{a}$.

(iv) $L = \prod_{i=1}^{n} f(x_i) = \left(\frac{1}{b}\right)^n$

$\ln L = n \ln(1/b)$

$\frac{d \ln L}{db} = \frac{n}{1/b} \cdot \frac{-1}{b^2}$

$= \cancel{\;} - \frac{n}{b} = 0$

~~Hence~~ $\frac{d^2 \ln L}{d b^2} = \frac{n}{b^2} > 0$

$\therefore \hat{b} = \max(x_i)$

1b] $H_0 : p = 0.1$

$H_1 : p > 0.1$

(i) P(Type I error) = P(rejecting H_0 | H_0 is true)

$= P(X \geq 3 \mid p = 0.1) + P(X \geq 5 \mid 0.1) \ldots$

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Public	24	45	29	33	20	40	26.5
7] Put	30	54.5	30	40	28.5	36	30.5

25	27.5
30.5	35.5

(i) $n = 9$

$\overline{X}_1 = 30$ $\overline{x}_2 = 35.055$

$S_1^2 = 64.3125$ $S_2^2 = 67.403$

(iii) $H_0: \mu_1 = \mu_2$ $S_p^2 = \frac{1}{16}[514.5 + 606.827]$

$H_1: \mu_1 \neq \mu_2$

$= 70.07$

$\therefore T.S = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{(70.0704)(0.2222)} \sim t_{n_1+n_2-2}$

$= \cancel{0.0}\ \cancel{0.322626}\ 1.3216$

CI at 95% LOS $= 2.12 \pm 0.3246$

$(1.7954, 2.4496)$

$t_{cal} < t_{tab}$ We reject H_0.

~~iii) H_0: ..~~ ii).

8] i) Unbiased estimator 7]

This is when bias is 0. i.e the estimators ~~es parameter's~~ expected value is equal to population parameter

(ii) Biased
is the measure of difference between the expected value of the estimator and the parameter being ~~esit~~ estimated

iii) $MSE(\hat{\theta}) = var(\hat{\theta}) + Bias^2$

iv) The estimator having less MSE is more consistent

(v) They both will be considered to be consistent when MSE minimizes such that $n \to \infty$ and $MSE \to 0$

(vi) We can estimate θ by method of moments. We can also use a more ~~preffered~~ preferred ~~mo~~ method of Maximum Likelihood Estimate.

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9]

i) The variable ~~χ~~ ~~$\bar{X}_k$~~ t_k is defined as

$$\frac{N(0,1)}{\sqrt{t_k/K}} \sim t_k$$

where,

t_k is the t distribution with 'K' degrees of freedom.

K is the number ~~&~~ of samples or simply 'n-1'.

ii) mean = 0

$$\text{variance} = \frac{K}{K-2}$$

iii) $n = 10$

$\bar{x}$ ~~$\bar{x}$~~ $= 50$

$s^2 = 48.667$

a) $\therefore \dfrac{\bar{x} - \mu}{s\sqrt{n}} \sim$ ~~t_{n-1}~~ t_{n-1}

~~$P(t_{9,0.5\%} < \bar{x}$~~ $P\left(t_{9,0.5\%} < \dfrac{\bar{x}-\mu}{s/\sqrt{n}} < t_{9,99.5\%}\right)$

$= 0.99$

$$TS = \bar{x} \pm 3.25 \times \sqrt{\frac{48.667}{10}} = 50 \pm 3.25(2.206)$$

$\therefore$ 99% CI = (42.83, 57.17)

iv) From (iii)

~~t_{n-1}~~ $t_{n-1} \sim N\left(0, \sqrt{\dfrac{n-1}{n-3}}\right)$

$$\frac{\bar{x}-\mu}{s/\sqrt{n}} \sim N(0, \sqrt{9/7})$$

$$T.S = \bar{x} - z_{0.5\%}\sqrt{s^2/n}$$

$$= 50 - 2.58\sqrt{\frac{48.667 \times 9}{10}}$$

99% CI = (43.54, 56.45)

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10] $n = 1000$ $\quad X \sim Poi(n\lambda)$

$\lambda = 3$ $\quad X \sim Poi(3000)$

$H_0 : n\lambda \geq \cancel{3000}\ 3100$

$H_1 : n\lambda \leq 3100$

i) P(Type 1 error) = P(Reject Ho | Ho is True)

$= P(X > 3100 \mid X \sim Poi(3000))$

$$P(X > 3100) = P\left(Z > \frac{3100 - 3000}{\sqrt{3000}}\right)$$

$$= P(Z > 1.8257)$$

$$= \underline{\underline{0.0343}}$$

ii) Type 2 error = event of accepting Ho when false

iii) Power of test = P(reject Ho | Ho is false)

$P[X > 3100 \mid X \sim Poi(n\lambda \div N(n\lambda, n\lambda)]$

The value of power of test depends on the value of the parameter under hypothesis

iv) $\hat{\mu}$ is the estimated value

$\therefore \hat{\mu} \sim N(\mu, \hat{\mu}/n)$ $\qquad$ 99% CI

$\dfrac{\hat{\mu} - \mu}{\sqrt{\hat{\mu}/n}} \sim N(0,1)$ $\qquad = (2.7613, 3.0387)$

$$P\left(-2.578 < \frac{2900 - \mu}{\sqrt{2900/100}} < 2.578\right) = 0.99$$

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Q11) $n = 12$ $\quad \Sigma x = 213200$ $\quad \Sigma x^2 = 4919860000$

$\bar{x} = 17767$ $\quad s = 10144$

new $\Sigma x_1 = 213200 - 11000 - 48000$
$+ 27500 + 31500$
$= 213200$

$\bar{x}_1 = \frac{213200}{12} = 17767$

$\Sigma x_2 = 4243360000$

$$s_1^2 = \frac{1}{n-1}\left[\Sigma x_i^2 - n(\bar{x}_1)^2\right]$$

$$= \frac{1}{11}(4243360000 - 3787995468)$$

$s^2 = 41396775.64$

There is no change in $\bar{x}$. But s^2 decreases. showing that new data is better sample as the var has reduced.

Q12) i) $X \sim U(0, \theta)$

The CRLB result holds under very general conditions except where the range of the dist. involves the parameter.

iii) Bias of estimator $\hat{\theta}(c)$ is given

$$\text{Bias}(\hat{\theta}(c)) = E[\hat{\theta}(c) - \theta]$$

$$= E[cY - \theta] \qquad \frac{(cn - 1)\theta}{n+1}$$

$$= c \cdot E[Y] - \theta$$

$$= c \cdot \frac{n\theta}{n+1} - \theta =$$

$$MSE[\hat{\theta}(c)] = E[(\hat{\theta}(c) - \theta)^2]$$

$$= E[(cY - \theta)^2]$$

$$= c^2 E[Y^2) - 2c\theta E(Y) + \theta^2$$

$$= c^2 \cdot \frac{n\theta^2}{n+2} - 2c\theta \cdot \frac{n\theta}{n+1} + \theta^2$$

$$= c^2 \cdot \frac{n\theta^2}{n+2} - c \cdot \frac{2n\theta^2}{n+1} + \theta^2$$

V) in order to minimize the MSE

$$0 = \frac{d}{dc} MSE[\hat{\theta}(c)]$$

$$c_m = \frac{n+2}{n+1} > 0 \quad \therefore \text{It is minimum.}$$