

FE Assignment 1

i) Arbitrage opportunity is a situation where we can make a certain profit with no risk. This is sometimes described as a free lunch

An arbitrage opportunity means that

(a) we can start at time 0 with a portfolio that has a net value of zero (implying that we are long in some assets and short in others). This is usually called a zero-cost portfolio.

(b) at some future time T:

the probability of a loss is 0

the probability that we make a strictly positive profit is greater than 0

If such an opportunity existed then we could multiply up this portfolio as much as we wanted to make as large a profit as we desired.

i) Law of one price

The Law of one price states that any two portfolios that behave in exactly the same way must have the same price. If this were not true, we could buy the 'cheap' one and sell the 'expensive one to make an arbitrage (risk-free) profit.

iii)

a) Using Put-Call parity, the value of put option should be:

$$p_1 = c_1 + K \exp(-(T-1)) - S \exp(-q(T-1))$$

$$-30 + 120 \exp(-0.05 \cdot 25) - 125 \exp(-0.15 \cdot 25) = -28.11$$

b) Arbitrage profit

If the put options are only Rs. 23 then they are cheap. If things are cheap then we

buy them.

So looking at the put-call parity relationship, we "buy the cheap side and sell the

expensive side",

ie we buy put options and shares and sell call options and cash.

For example:

sell 1 call option Rs. 30

buy 1 put option (Rs. 23)

buy 1 share (Rs.125)

sell (borrow) cash Rs.118

This is a zero-cost portfolio and, because put-call parity does not hold, we know it will make an arbitrage profit. We can check as follows: TATIVE STUDIES

In 3 months' time, repaying the cash will cost $118 \exp(0.05 \cdot 3/12)$ Rs. 119.48

We also receive dividends d on the share.

1. If the share price is above 120 in 3 months' time then the other party will exercise their call option and we will have to give them the share. They will pay 120 for it and our profit is:

$$120 - 119.48 + d = 0.52 + d$$

(The put option is useless to us)

2. If the share price is below 120 in 3 months' time then we will exercise our put option and sell it for 120. Our profit is: $120 - 119.48 + d = 0.52 + d$

(The call option is useless to the other party and will expire worthless)

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Let t_1 and t_2 be ex-dividend dates are anticipated for a stock and t be the times before which the stock goes ex-dividend. Dividends are denoted by d_1 and d_2 .

If the option is exercised prior to the first ex-dividend date then the investor receives $S(t) - K$. If the option is not exercised, the price drops to $S(t) - d_1$. The value of the American option is greater than $S(t) - K$ if $S(t) - K > S(t) - d_1$. The option is never optimal to exercise the option if $S(t) - K < S(t) - d_1$.

Using this equation we have $K(1 - e^{-rt}) = 350(1 - e^{-0.05 \times 0.5}) = 18.87$

and $65(1 - e^{-0.05 \times 0.5}) = 10.91$. Hence it is never optimal to exercise the American option on the two ex-dividend dates.

5. The required probability is the probability of the stock price being greater than Rs. 258 in 6 months' time.

The stock price follows Geometric Brownian motion. Let $S(t) = S(0) \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$. Therefore $\ln(S(t))$ follows normal distribution with mean $\ln(50) + \left(\mu - \frac{\sigma^2}{2}\right)t$ and variance $(\sigma^2)t$.

Implies $\ln(S(t))$ follows $N\left(\ln(50) + \left(0.16 - \frac{0.35^2}{2}\right) \times 0.5, 0.35 \times 0.5^{1/2}\right) = N(5.59, 0.247)$.

This means $(\ln(S(t)) - 5.59) / 0.247^{1/2}$ follows standard normal distribution,

Hence the probability that stock price will be higher than the strike price of Rs. 258 in 6 months time $= 1 - N(5.55 - 5.59) / 0.247^{1/2} = 1 - N(-0.1364) = 0.5542$.

The put option is exercised if the stock price is less than Rs. 258 in 6 months time.

The probability of this $1 - 0.5542 = 0.4457$.

6.

i) The given relationship can be written as:

Since S_t is a function of standard Brownian motion, B_t , applying Ito's Lemma, the SDE for the underlying stochastic process becomes:

$$dB_t = 0 \cdot dt + 1 \cdot dB_t$$

Let $G(t, B) = S$, then $dG/dt = u$ and $dG/dB = \sigma$

$$dG/dt = u$$

$$dG/dB = \sigma$$

Hence, using Ito's Lemma from Page 46 in the Tables we have: $dG = \frac{\partial G}{\partial t} dt + \frac{\partial G}{\partial B} dB + \frac{1}{2} \frac{\partial^2 G}{\partial B^2} dt$

$$\text{Le. } ds = \left(\frac{\partial S}{\partial t} + \frac{1}{2} \sigma^2 \right) dt + \sigma dB$$

Thus,

$$ds/S = \left(\frac{\partial S}{\partial t} + \frac{1}{2} \sigma^2 \right) \frac{dt}{S} + \sigma \frac{dB}{S}$$

$$\text{So, } \frac{\partial S}{\partial t} = 0 \text{ and } \frac{\partial^2 S}{\partial B^2} = \frac{1}{2} \sigma^2$$

ii)

The expected value of S , is:

$$E[S] = E[S] \gg S \mu e^{3a} E[e^{***}] -$$

Since $B_N(0,1)$, its MGF is $E[e^x] = 21 e^{So}$, $E[S] - So$ etxe% Soe % olt

The variance of S . is:

$$\text{Var}[S] = E[S^2] - (E[S])^2$$

$$= E[S^2 e^2 + 208] - (So e^{15-0212})$$

$$= S^{3/4} e^{2a} (e^{dsht} \dots 802)$$

$$\text{Cov}[S_1, S_2] = E[S_1, S_2] - E[S_1] E[S_2]$$

From above,

$$E[S] \text{ So ease and } E[Su] \text{ So et}\%22$$

The expected value of the product is:

$$E[Su \cdot Su] = E[S, \text{explut: } +a Bu] \text{ So exp (uta a Bu)}$$

$$Se E \exp(aB_1 + aBu)$$

To evaluate this we need to split Bu into two independent components: Buz But + $(Biz - Bu)$ where $Buz - Bu \sim N(0, t_2 - t_1)$

Hence,

$$= S^2 e^{+2} E \exp(a8u + a(Ba + (Ba - Bu))) = S^2 e^2 E \exp(208_1 + 0(B - Bu))$$

$$= S^2 e^2 E[\exp(20Ba)] E \exp(Buz - Ba) = S^2 e^2 \exp(20^3 ts) \exp[-\frac{1}{2} \sigma^2 (t_2 - t_1)]$$

7.

i) Setting up the commodity tree using u for up move and d for down move, p is up-step probability.

Where p is the up probability and $(1-p)$ the down probability

Then $E(C) = S(pu + (1-p)d)$, and

$$\text{Var}(C) = E(C^2) - E(C)^2$$

$$= S^2 [p^2 u^2 + (1-p)^2 d^2 - 2p(1-p)ud]$$

$$= S^2 [p^2 u^2 + (1-p)^2 d^2 - 2p(1-p)ud]$$

$$= S^2 [p^2 u^2 + (1-p)^2 d^2 - 2p(1-p)ud]$$

$$= S^2 [p^2 u^2 + (1-p)^2 d^2 - 2p(1-p)ud]$$

C)

c) The lookback call pays the difference between the minimum value and the final value.

Notate paths by U for up and D for down, in order

We get the payoffs

UUU

(100 186-80-20.186 UDU (86 232-80-6.232

Node A

Node B

UUD (36 232-80) 6.232 UDD (7422-74 22)-0

Node B Node C

DUU (86 232-74 22)-12.012 Node B

DUD

(74 22-74 22)-0

Node C

DDU

(74.22-68.857)-5.363 DDD (63.882-63.882)

Node C

Node D

The lookback payoffs are, for each successful path (i.e. with a non-zero result)

Probabilities of arriving at each node are:

Node A p 11147

Node 8 $p/(1-p) = 12015$ Node C- $p(1-p-12950$

Node D $p(1-p-13959$

Hence the tree value of lookback option is:

$(11147 \cdot 20.186) + (12015 \cdot (6.232 + 6.232 + 12.012)) + (12950 \cdot 5.363)$

-5.8854

i Consider a stock whose current price is S_0 and an option whose current price is f . We suppose that the option lasts for time T and that during the life of the option the stock price can either move up from S_0 to a new level S_u or move down to S_d where $u > 1$ and $d < 1$.

Let the payoff be f_u if the stock price becomes S_u and f_d if stock price becomes S_d . Let us construct a portfolio which consists of a short position in the option and a long position in A shares. We calculate the value of A that makes the portfolio risk-free. Now if there is an

upward movement in the stock the value of the portfolio becomes $AS_u - f_u$ and if

there is a downward movement of stock, the value of the portfolio becomes $AS_d - f_d$

The two portfolios are equal if $AS_u - f_u = AS_d - f_d$

Or $A = \frac{f_u - f_d}{S_u - S_d}$

interest

so that the portfolio is risk-free and hence must earn the risk free rate of

This means the present value of such a portfolio is $(AS_u - f_u) \exp(-rT)$

Where r is the risk free rate of interest.

The cost of the portfolio is $AS_0 - f$

Since the portfolio grows at a risk free rate, it follows that

ii. The option pricing formula does not involve probabilities of stock going up or down although it is natural to assume that the probability of an upward movement in stock increases the value of call option and the value of put option decreases when the probability of stock price goes down.

This is because we are calculating the value of option not in absolute terms but in terms of the value of the underlying stock where the probabilities of future movements (up and down) in the stock already incorporates in the price of the stock. However, it is natural to interpret p as the probability of an up movement in the stock price.

The variable $1-p$ is then the probability of a down movement such that the above equation can be interpreted as that the value of option today is the expected future value discounted at the risk free rate

The expected stock price $E(S)$ at time T - $pS_{uu} + (1-p)S_{ud}$ $\text{Sed } 0.5$

or $E(S) = pS_{eu-d} + S_{ud} - 0.5$

Substituting p from above equation in (i) Le. $p = \frac{e^{-rt}(S_{ud} - S)}{S_{eu-d} - S}$

We get $E(S) = e^{-0.5-1}$ Le. stock price grows at a risk free rate or return on a stock is risk free rate

iv. In a risk neutral world individuals do not require compensation for risk or they are indifferent to risk. Hence expected return on all securities and options is the risk free interest rate. Hence value of an option is its expected payoff in a risk neutral discounted at risk free rate.

UNIT 2

9.

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The forward price is given by $FS = S \exp(rt)$ where S is the stock price, t is the delivery time and r is the continuously compounded risk-free rate of interest applicable up to time

t

Put-call parity states that $c + K \exp(-rt) = p + S$ where c and p are the prices of

a European call and put option respectively with strike K and time to expiry t and S is the current stock price.

To compute F , we need to find S and $r.t$ is given to be 0.25 years.

Substituting the values from the first two rows of the table in the put-call parity, we get two equations in two unknowns (S and r):

$$13.334 + 70 - \exp(-0.25r) = 0.120 + S$$

$$8.869 + 75 - \exp(-0.25r) = 0.568 + S$$

Solving the simultaneous equations for S and r, we get

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5-82 and $r=7\%$

Therefore, we get the forward price $F = 82 \exp(0.07 \cdot 0.25) = 83.45$

(4)

Let the (continuously compounded, annualized) rate of interest over the next k months

be n. Then the required forward rate r_e can be found from:

$$\exp(r_e \cdot 0.5) = \exp(r \cdot 0.25) \exp(r \cdot 0.25) \text{ or } 2$$

We know that $r, 7\%$

11.

i) A recombining binomial tree or binomial lattice is one in which the sizes of the up-steps and down-steps are assumed to be the same under all states and across all time intervals, ie, $u_t = u$ and $d_t = d$ for all times t and states j , with $d < \exp(r) < u$

It therefore follows that the risk neutral probability 'q' is also constant at all times

and in all states eg. $q_t = q$ The main advantage of a 'n' period recombining binomial tree is that it has only $[n+1]$ possible states of time as opposed to 2^n possible states in a similar non recombining binomial tree. This greatly reduces the amount of computation time required when using a binomial tree model.

The main dis-advantage is that the recombining binomial tree implicitly assumes that the volatility and drift parameters of the underlying asset price are constant over time, which assumption is contradicted by empirical evidence

a) The risk-neutral probabilities at the first and second steps are as follows:

$$q_1 = (\exp(0.0175) - 0.95) / (1.10 - 0.95)$$

$$(0.067651 / 0.15)$$

$$= 0.4510$$

$$q_2 = (\exp(0.025) - 0.901) / (1.1 - 0.90)$$

$$= 0.41772$$

Put payoffs at the expiration date at each of the four possible states of expiry are 0, 0, 0 and 95.

Working backwards, the value of the option V1 (1) following an up step over the first 3 months is

$$V1(1) = \exp(0.025) \cdot (0.41772 \cdot 0 + 0.58228 \cdot 10.58228) \text{ i.e., } V1(1) = 10.58228$$

The value of the option V1 (2) following a down step over the first 3 months is: $V1(2) = \exp(0.025) \cdot (0.41772 \cdot 10.58228 + 0.58228 \cdot 95)$ Le, $V1(2) = 53.9508$

The current value of the put option is: $VO = \exp(0.0175) \cdot (0.4510 \cdot 10.58228 + 0.5490 \cdot 53.9508)$ Le, $VO = 29.105$

b) While the proposed modification would produce a more accurate valuation, there would be a lot more parameter values to specify. Appropriate values of u and d values of r for each month would be required for each branch of the tree and values of r for each month would be required.

The new tree would have $2^6 = 64$ nodes in the expiry column. This would render the calculations prohibitive to do normally, and would require more programming and calculation time on the computer. An alternative model that might be more efficient numerically would be a 6-step recombining tree which would have only 7 nodes in the final column.

12.

12.

Given $Z(t)$ is standard Brownian a. $du(t) = 2d^2(t) - 0$

$-0dt + 2dz(t)$

Thus, the stochastic process $(U(t))$ has zero drift.

b. $dv(t) = d[2(t) - dt]$

$d[2(t) - 2Z(t)dz(t) + \frac{1}{2}(dz(t))^2]$

$= 2(1)dz(t) + dt$ by the multiplication rule

Thus, $dv(t) = 2Z(t)dz(t)$. The stochastic process $(V(t))$ has zero drift.

c. $dW(t) = d[tZ(t) - \frac{1}{2}t^2dt]$

Because $d[tZ(t)] = 1dz(t) + 2tZ(t)dt$, we have

$dW(t) = t dz(t)$.

Thus The process $(W(t))$ has zero drift

13.

13.

L

Let S/S , follows lognormal distribution with parameters $(u - \frac{1}{2}\sigma^2)t$, and $\sigma\sqrt{t}$ such the

expected return on a stock is u and volatility is σ . This means Expected value of stock price at the end of first time step = $S_0 e^{u\Delta t}$. On the tree the expected price at that time = $qS_{u\Delta t} + (1-q)S_{d\Delta t}$

In order to match the expected return on the stock with the tree's parameters we have

$$qS_{u\Delta t} + (1-q)S_{d\Delta t} = S_0 e^{u\Delta t}$$

$$q = \frac{e^{u\Delta t} - d}{u - d}$$

Volatility σ of a stock price is defined so that $\sigma\sqrt{t}$ is the standard deviation of the return on the stock price in a short period of time t

$$\text{The variance of stock price return is } q^2 + (1-q)^2 - [qu + (1-q)d] = \sigma^2 \Delta t$$

Substituting the value of q in the expression above we have

$$(u+d) - ud - e^{u\Delta t} = \sigma^2 \Delta t$$

When higher powers of Δt other than Δt are ignored. This implies $u - e^{u\Delta t} = -\frac{1}{2}\sigma^2 \Delta t$ and $d = 1 - \sigma^2 \Delta t$