SRM-II Assignment 2

1)(i)

$$\begin{array}{c|cc} & B & O \\ \hline B & -\lambda & \lambda \\ O & \mu & -\mu \end{array}$$

(ii) the holding times in each state follows exponential distribution. $B \sim exp(\lambda)$, $O \sim exp(\mu)$

(iii)
$$\frac{d}{dt}\,{}_tP_s^{BB} = {}_tP_s^{BB}\cdot\mu_{BB} + {}_tP_s^{BO}\cdot\mu_{OB}.$$
$$= {}_tP_s^{BO}\cdot\mu - {}_tP_s^{BB}\cdot\lambda. //$$

$$\frac{d}{dt}\,{}_tP_s^{BO} = {}_tP_s^{BB}\cdot\mu_{BO} + {}_tP_s^{BO}\cdot\mu_{OO}.$$
$$= {}_tP_s^{BB}\cdot\lambda - {}_tP_s^{BO}\cdot\mu \quad //$$

(iv) ${}_tP_s^{BO} = 1 - {}_tP_s^{BB}$ $\Rightarrow$ as it is a two-state model.

$$\frac{d}{dt}\,{}_tP_s^{BB} = (1 - {}_tP_s^{BB})\cdot\mu - {}_tP_s^{BB}\cdot\lambda$$

$$\frac{d}{dt}\,{}_tP_s^{BB} + (\mu+\lambda)\,{}_tP_s^{BB} = \mu \;;\quad IF = e^{\int(\mu+\lambda)dt} = e^{(\mu+\lambda)t}$$

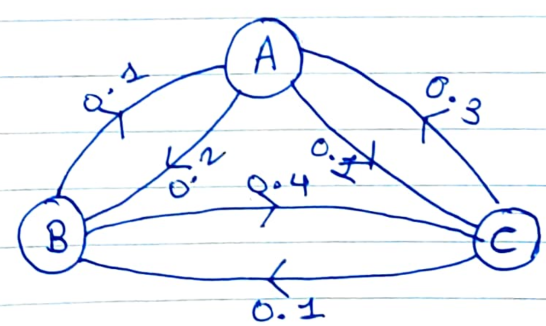
$${}_tP_s^{BB} = \frac{1}{e^{(\mu+\lambda)t}}\left[\int e^{(\mu+\lambda)t}\cdot\mu\, dt\right]$$

$$e^{(\mu+\lambda)t}\cdot{}_tP_s^{BB} = \frac{\mu}{\mu+\lambda}\cdot e^{(\mu+\lambda)t} + C$$

at $t=0$, in state B, so $C = \frac{\cancel{\mu}\,\lambda}{\mu+\lambda}$ //

so $${}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu}\cdot e^{-(\lambda+\mu)t} \quad //$$

2)(i)



(ii) $$\frac{d}{dt}P_{AA}(t) = P_{AA}(t)\cdot\mu_{AA} + P_{AB}(t)\cdot\mu_{BA} + P_{AC}(t)\cdot\mu_{CA}$$
$$= 0.1\,P_{AB}(t) + 0.3\,P_{AC}(t) - 0.3\,P_{AA}(t)$$

$$\frac{d}{dt} P_{AB}(t) = P_{AA}(t)\cdot \mu_{AB} + P_{AB}(t)\cdot \mu_{BB} + P_{AC}(t)\cdot \mu_{CB}$$

$$= 0.2\, P_{AA}(t) + 0.1\, P_{AC}(t) - 0.5\, P_{AB}(t)$$

$$\frac{d}{dt} P_{AC}(t) = P_{AA}(t)\cdot \mu_{AC} + P_{AB}(t)\cdot \mu_{BC} + P_{AC}(t)\cdot \mu_{CC}$$

$$= 0.1\, P_{AA}(t) + 0.4\, P_{AB}(t) - 0.4\, P_{AC}(t)$$

(iii) λ_A (total rate going out of A) $= 0.2 + 0.1 = 0.3$

$$= e^{-\int_0^2 0.3\, dt} \Rightarrow e^{-[0.3t]_0^2} \Rightarrow e^{-0.6} \Rightarrow \underline{0.5488}$$

~~(iv)~~

~~A → A → A → C~~

~~A → A → B → C~~

~~A → A → C → C~~ ✗

~~A → B → A → C~~ ✓

~~A → B → B → C~~ ✓

~~A → B → C → C~~ ✗

~~A → C → A → C~~

~~A → C → B → C~~ ✓

~~A → C → C → C~~ ✗

$$\cancel{\begin{bmatrix} -0.3 & 0.2 & 0.1 \\ 0.1 & -0.5 & 0.4 \\ 0.3 & 0.1 & -0.4 \end{bmatrix} \begin{bmatrix} -0.3 & 0.2 & 0.1 \\ 0.1 & -0.5 & 0.4 \\ 0.3 & 0.1 & -0.4 \end{bmatrix}}$$

$$\cancel{= \begin{bmatrix} 0.14 & -0.15 & 0.01 \\ 0.04 & & \\ & & \end{bmatrix} \begin{bmatrix} -0.3 & 0.2 & 0.1 \\ 0.1 & -0.5 & 0.4 \\ 0.3 & 0.1 & -0.4 \end{bmatrix}}$$

3) ~~(i)~~

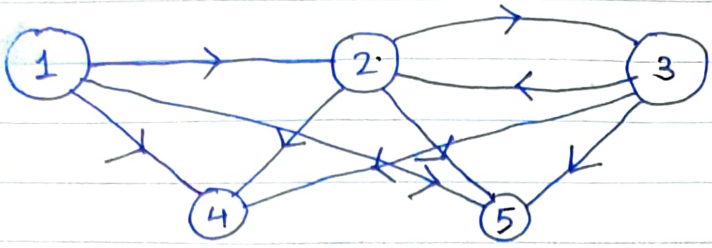
(iv) $A \to B \to A \to C \Rightarrow \frac{2}{3} \times \frac{1}{5} \times \frac{1}{3} = \frac{2}{45}$

$A \to C \to B \to C \Rightarrow \frac{1}{3} \times \frac{1}{4} \times \frac{4}{5} = \frac{1}{15}$

~~A~~ $A \to C \to A \to C \Rightarrow \frac{1}{3} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{12}$.

Prob $= \frac{7}{36} = \underline{\underline{0.194}}$

3)(i)



① Never taken Nimble, ② Taking Nimble, ③ No longer taking Nimble, ④ Dead through heart disease, ⑤ Dead through other causes.

(ii)

$$ {}_{dt+t}p_x^{34} = {}_tp_x^{31}\,{}_{dt}p_{x+t}^{14} + {}_tp_x^{32}\,{}_{dt}p_{x+t}^{24} + {}_tp_x^{33}\,{}_{dt}p_{x+t}^{34} $$
$$ + {}_tp_x^{34}\,{}_{dt}p_{x+t}^{44} + {}_tp_x^{35}\,{}_{dt}p_{x+t}^{54} $$

$$ {}_{dt+t}p_x^{34} = {}_tp_x^{32}\,{}_{dt}p_{x+t}^{24} + {}_tp_x^{33}\,{}_{dt}p_{x+t}^{34} + {}_tp_x^{34}\,\underbrace{{}_{dt}p_{x+t}^{44}}_{\Downarrow\ 1} $$

$$ {}_{dt}p_{x+t}^{ij} = \mu_{x+t}^{ij}\,dt + o(dt). $$

$$ {}_{dt+t}p_x^{34} = {}_tp_x^{32}\,\mu_{x+t}^{24}\,dt + {}_tp_x^{33}\,\mu_{x+t}^{34}\,dt $$
$$ + {}_tp_x^{34} + o(dt) $$

$$ {}_{dt+t}p_x^{34} - {}_tp_x^{34} = {}_tp_x^{32}\,\mu_{x+t}^{24}\,dt + {}_tp_x^{33}\,\mu_{x+t}^{34}\,dt + o(t $$

4)(i) expected number of calls per hour is 3

(ii) $(1/3)$ hours $=$ 20 minutes

(iii) $= e^{-1.5} = \underline{0.22313}$

(iv) 3 calls per hr so $3 \times 7 = 21$ min in 1 hr spend talking on call so prob busy $\Rightarrow \frac{21}{60} = \underline{\underline{0.35}}$

5)(i) $$ \frac{d}{dt} P_{HH}(x,t) = \lim_{h \to 0} \frac{P_{HH}(x,t+h) - P_{HH}(x,t)}{h} $$

$$ P_{HH}(x,t+h) = P_{HH}(x,t)\cdot P_{HH}(t, t+dt) $$
$$ + P_{HS}(x,t)\,P_{SH}(t, t+dt) + P_{HD}(x,t)\,P_{DH}(\cancel{t}, t+dt) $$

$$ P_{HH}(x, t+dt) = P_{HH}(x,t)\cdot P_{HH}(t,t+dt) + P_{HS}(x,t)\cdot P_{SH}(t,t+dt) $$

$$p_{ij}(t, t+dt) = \lambda_{ij}(t)\,dt + o(dt).$$
$$p_{ii}(t, t+dt) = 1 + \lambda_{ii}(t)\,dt + o(dt)$$

so $P_{HH}(x, t+dt) = P_{HH}(x,t)\,(1 - \sigma(t)dt - \mu(t)dt)$

$+ P_{HS}(x,t)\,\rho(t, C_t) + o(dt)$

$$\frac{d}{dt} P_{HH}(x,t) = P_{HH}(x,t)\,(-\sigma(t) - \mu(t)) + P_{HS}(x,t)\,\rho(t, C_t).$$

(ii) so $\frac{d}{dt} P_{\overline{HH}}(0,t) = -(\sigma(t) + \mu(t)) \cdot P_{\overline{HH}}(t)$

$$\left[\ln P_{\overline{HH}}(0,t)\right]_0^t = \int_0^t -(\sigma(s) + \mu(s))\,ds$$
dummy

$$P_{\overline{HH}}(0) = 1$$

$$P_{\overline{HH}}(0,t) = e^{-\int_0^t [\sigma(s) + \mu(s)]\,ds}$$

6) Similarities :

All have discrete state space, All have the markov property

Differences:

Markov chain & jump chain have discrete time domain but markov ~~jump~~ jump process has continuous time domain.

Markov chain & jump chain are expressed as probablities but markov jumps are expressed in transition rates.

Markov chains have loops in each state, markov jump chain have loops in only absorbing states, markov jumps has no loops.

7) (i) the maximum likelihood estimates of transition intensity from i to j is no of transitions from i to j divided by total waiting time in i. Find ~~ext~~ total time spent in each state.

(ii) $$\frac{d}{dt} P_{AA}(s,t) = - P_{AA}(s,t) \cdot \mu$$

$$\frac{d}{dt} P_{AT}(s,t) = P_{AA}(s,t) \cdot \mu + P_{AT}(s,t) \cdot 0 = P_{AA}(s,t) \cdot \mu$$

~~(iii)~~ $$\cancel{P_{AA}(t) = e^{-\mu t}} \quad \cancel{\Rightarrow \text{as only 2 states.}}$$

$$\cancel{\frac{d}{dt} P_{AT}(t) = (1 - P_{AT}(t)) \cdot \mu}$$

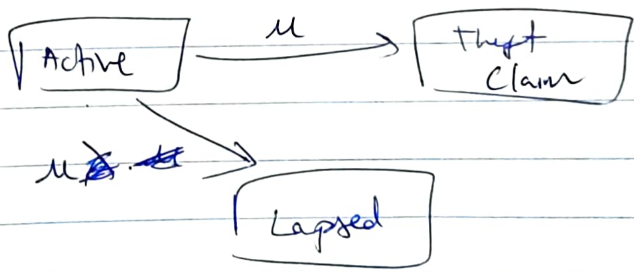
$$\cancel{\frac{d}{dt} P_{AT}(t) + \mu \cdot P_{AT}(t) = \mu} \quad \cancel{IF = e^{\mu t}}$$

$$\cancel{P_{AT}(t) =}$$

(iii) $$P_{AA}(t) = e^{-\mu t}$$

$$P_{AT}(t) = 1 - e^{-\mu t} \quad \Rightarrow \text{as only 2 states}$$

(iv)



(v) $$\frac{d}{dt} P_{AA}(t) = - P_{AA}(t)(2\mu)$$

$$\frac{d}{dt} P_{AA}(t) + (2\mu) P_{AA}(t) = 1$$

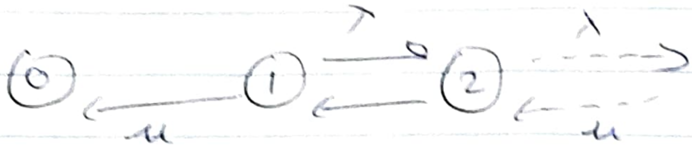
$$P_{AA}(t) = \frac{1}{e^{2\mu t}} \left[\int e^{2\mu t} \cdot 1 \, dt + c \right]$$

$$\frac{1}{e^{2\mu t}} \cdot \left[\frac{e^{2\mu t}}{2\mu} + c \right] = P_{AA}(t)$$

$$= \frac{1}{e^{2\mu t}} \cdot \left[\frac{e^{2\mu t}}{2\mu} + 1 - \frac{1}{2\mu} \right] \times C$$

8) (i) S 0, 1, 2, 3 ....

(ii)



(iii)
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdot \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 & \\ 0 & \mu & -(\mu+\lambda) & \lambda & 0 & \\ 0 & 0 & \mu & -(\mu+\lambda) & \lambda & \\ 0 & 0 & 0 & \mu & -(\mu+\lambda) & \end{bmatrix}$$

(iv) Jump chain is each distinct state visited in the order visited where the time set is the times when states are moved between.

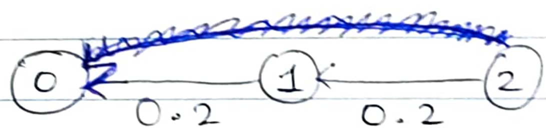
(v)
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdot \\ \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 & 0 & \\ 0 & \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 & \\ 0 & 0 & \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & \\ \vdots & 0 & 0 & \mu/(\mu+\lambda) & 0 & \cdot \end{bmatrix}$$

(vi) $\left[\dfrac{\mu}{\mu+\lambda}\right]^3$

9) (i) sum of rows = 0.
waiting time follows exp. dist.

(ii) S 0, 1, 2.

(iii)



(iv) $$\frac{d}{dt} P_{22}(t) = -0.2\,P_{22}(t).$$

$$\frac{d}{dt} P_{21}(t) = 0.2\,P_{22}(t) - 0.2\,P_{21}(t).$$

$$\frac{d}{dt} P_{20}(t) = 0.2\,P_{21}(t)$$

10)(i) $e^{-\int_0^s \cancel{2}t^2 \, dt} \Rightarrow e^{-\cancel{2}s^2}$ [~~tanks~~ seems to be printing mistake in Q10]

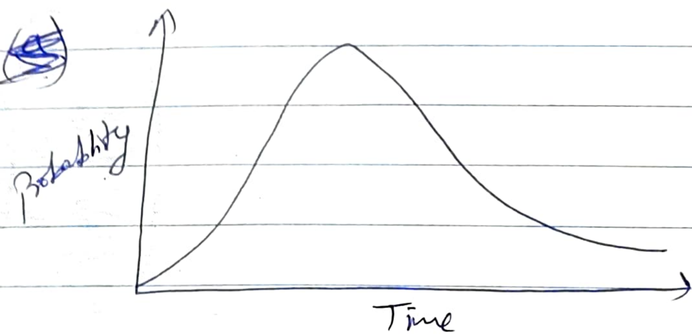
(ii) $= \int P_{\overline{AA}}(s) \times 2s \times P_{\overline{BB}}(s,T)\, ds.$

using kolmogrov integral

$$= \int_0^t \cancel{e^{-s^2}} \times 2s \times e^{-(T^2 - \cancel{s^2})} \, ds.$$

$$= \int_0^t 2s \times e^{-T^2} \, ds \Rightarrow e^{-T^2} \times T^2.$$

(iii)



Prob inc from time 0 to T as transition rates from A to B inc, as transitions increase, process visiting state B becomes more probable, decreasing ~~probablity~~ probablity.

$$\text{func} = e^{-t^2} \times t^2 \quad \text{on}.$$

$$e^{-t^2} \times -2t \times t^2 + 2t \times e^{-t^2} = 0.$$

$$\cancel{2t}\,\cancel{e^{-t^2}} = 2t\, \cancel{e^{-t^2}} \times t^2 \qquad \boxed{t=1} \text{ or } t=0 \text{ not pos}$$

11)(i) N_t = no of claims upto time t

$$P(T_0 \le y \mid N_s = 1) = \frac{P(T_0 \le y, N_s = 1)}{P(N_s = 1)}$$

$$= \frac{P(N_y = 1, N_s - N_y = 0)}{P(N_s = 1)}$$

so $\dfrac{(\lambda y e^{-\lambda y})\, e^{\lambda(s-y)}}{\lambda s e^{-\lambda s}} = \dfrac{y}{s}$ $\therefore$ dis of ~~ede~~ uniform $[0,s]$.

(ii) $\lambda^n e^{-\lambda\left(\sum_{i=1}^{n} t_i\right)} \cdot 1$

(iii) $P(N_s = k \mid N_t = n) = \dfrac{P(N_s = k,\ N_t = n)}{P(N_t = n)}$

$= \dfrac{P(N_s = k,\ N_t - N_s = 0)}{P(N_t = n)}$

poisson process has stationary & independent increments,

$\Rightarrow \dfrac{e^{-\lambda s}\dfrac{(\lambda s)^k}{k!} \cdot \dfrac{e^{-\lambda(t-s)} \cdot \lambda^{n-k}(t-s)^{n-k}}{(n-k)!}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}}$

$= \dfrac{n!}{k!(n-k)!} \cdot \dfrac{s^k (t-s)^{n-k}}{t^k t^{n-k}}$

$= \binom{n}{k} \left(\dfrac{s}{t}\right)^k \left(1 - \dfrac{s}{t}\right)^{n-k}$

bin $\sim \left(n, \dfrac{s}{t}\right)$ ✓