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Roll Number: 413

Financial Engineering

Assignment 1

Unit 1 & 2

1.

(i) Arbitrage opportunity is a situation where we can make a certain profit without any risk.
Arbitrage opportunity also means that we start at time 0, with portfolio that has net value of 0, i.e. a zero-cost portfolio.
At some future time, T,
Probability of loss is zero and probability we make a strictly positive profit is greater than zero.

(ii) Law of one price states that "Assuming principal of no arbitrage, two securities or combinations of securities that gives exact same payments must have the same price. "

(iii) Pt: Price of 3 month European put
S0 =125
K = 120
c =30
Rf = 5% pa

(a) If dividends are paid continuously at rate of q=15%
Using put call parity,
ct + K$e^{-r(T-t)}$ = pt + St$e^{-q(T-t)}$
30 + 120$e^{-0.05 * 3/12}$ = pt + 125 $e^{-0.15*3/12}$
pt = 28.11

(b) If the price of put option is Rs. 23 then, it is cheaper.
To make profit, we buy cheap and sell the expensive. So, we will
buy put option @ Rs.23
Sell call option @ Rs.30
Buy one share @125
Borrow(sell) cash 118

So, in 3 months' time, cash to be re-payed would be 118*e^0.05*3/12 = 119.48
We will receive dividends on share, d

So, if share price is above 120, call option would be exercised by the holder and we would let our put option expire worthless.
Our profit payoff would be:
120 – 119.48 +d = 0.52 + d

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If share price is below 120, we would exercise our put option, and the holder of call option would let it go worthless. Our profit payoff would be:
120 – 119.78 + d = 0.52 + d

In either case, we generate positive profit at outset of 0.52

2. dXt = αμ(T-t) dt + σ √(T-t) dZt
f(x,t) = $e^{(m(T-t)-x)}$; where m is a differentiable equation
find ∂m/∂t if f(x,t) is a martingale

Ans.
$\frac{\partial F}{\partial t}$ = $e^{(m(T-t)\,-\,Xt)}*\frac{\partial m}{\partial t}$ = f($\frac{\partial m}{\partial t}$(T-t) +m)
$\frac{\partial F}{\partial dXt}$ = $e^{(m(T-t)\,-\,Xt)}*$ (-1) = -f
$\frac{\partial F^2}{\partial^2 dXt}$ = $e^{(m(T-t)\,-\,Xt)}$ = f

Therefore,
By ito's lemma,
df(x,t) = dXt $\frac{\partial F}{\partial t}$ + dt $\frac{\partial F}{\partial dXt}$ + $\frac{(dXt)^2}{2\,!}\frac{\partial F^2}{\partial^2 dXt}$
df = (αμ(T-t) dt + σ √(T-t) dZt)(-f) + dt f($\frac{\partial m}{\partial t}$(T-t) +m) + $\frac{\sigma^2(T-t)dt}{2}$ f
df = dt(-f αμ(T-t) + f ($\frac{\partial m}{\partial t}$(T-t) +m) + $\frac{\sigma^2(T-t)dt}{2}$ f) -f σ √(T-t) dZt
Since f(x,t) is a martingale, it means that it has zero drift.
i.e. dt(-f αμ(T-t) + f $\frac{\partial m}{\partial t}(\mathrm{T}-\mathrm{t})\ +\mathrm{m}$ + $\frac{\sigma^2(T-t)dt}{2}$ f) = 0
dt f (αμ(T-t) + $\frac{\partial m}{\partial t}$ (T-t) +m + $\frac{\sigma^2(T-t)dt}{2}$) = 0
$\frac{\partial m}{\partial t}$ = - (αμ(T-t) + $\frac{\sigma^2(T-t)dt}{2}$ $+m)*\frac{1}{(T-t)}$

3. dXt/Xt = 0.25 dt + σdWt
dXt = Xt(0.25dt + σdWt)
Yt = f(t,Xt)
Where f(t,Xt) = $e^{-t}x^2$
(i) Expression for dYt:
By ito's lemma.
dYt = dXt* $e^{-t}2Xt$ + dt* $e^{-t}x^2$*(-1) + $\frac{(dXt)^2}{2}$ * $2e^{-t}$
dYt =dXt* $e^{-t}2Xt$ - dt* $e^{-t}x^2$* + $\sigma^2\ dt\ Xt^2 * e^{-}t$
dYt = $2e^{-t}$ Xt^2 $\frac{dXt}{Xt}$ - Yt dt + Yt dt σ^2
dYt = 2Yt (0.25 dt + σdWt) - Yt dt + Yt dt σ^2
$\frac{dYt}{Yt}$ = (0.5 -1 + σ^2)dt + 2σdWt

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dYt = ($\sigma^2 \ - 0.5)$dt Yt + 2σdWt Yt

(ii) The process is martingale when the drift would be zero, i.e.
$\sigma^2 \ - 0.5$ = 0
σ^2 = 0.5

4. American call option
T= 10 months
Rf = 9.5% pa
K = 350
Dividend = 10 after three months from now and again after six months from now.
If option is exercised before prior to dividend date, then the payoff is (St-K).
After dividend,
Doubt

5. St follows geometric Brownian motion
St = So$e^{\ (\mu - \sigma^2/2)T + \sigma Xt}$
Therefore, ln(st) follows Normal distribution with [ln(So) + $(\mu - \sigma^2/2)T\ , \sigma^2 t)$
E(St) = 16%
Sd(St) = 35%
S0 = 254
In this case, mean = 5.5867
Std dev = 0.24748
Probability that European call with k=258 in 6 months will be exercised
Call will be exercised if the share price is above the strike price
Therefore,
P(St > 258) = P(ln(St) > ln(258)) = P(ln(St) > 5.553)

P($\frac{ln(St) - 5.5867}{0.24748}$ > $\frac{5.553 - 5.5867}{0.24748}$)

P(z > -0.136) = P(z < 0.136) = 0.55172

Probability that European put with same exercise price ad maturity will be exercised :
European put will only be exercised if share price is less than strike price

P(St<258) = 1 – 0.55172 = 0.4482

6. GBM
(i) Show that $\frac{dSt}{St} \ = \ xdBt \ + \ ydt$
Log($\frac{St}{So}$) = μt + σ Bt
St = So $e^{\mu t + \sigma \, Bt}$

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$\frac{\partial G}{\partial t}$ = μ* So $e^{\mu t + \sigma Bt}$ = μ St
$\frac{\partial G}{\partial St}$ = σ* So $e^{\mu t + \sigma Bt}$ = σ St
$\frac{\partial G^2}{\partial^2 t}$ = σ^2* So $e^{\mu t + \sigma Bt}$ = σ^2St

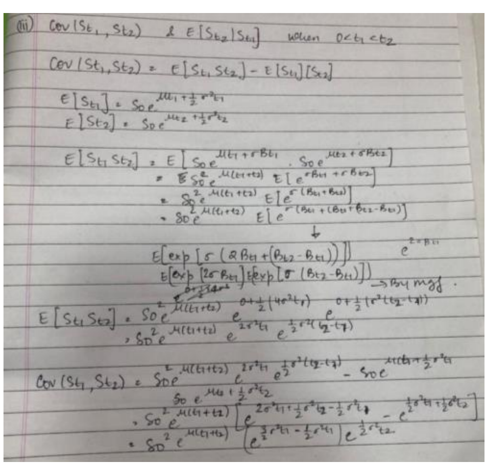
By ito's lemma,
dSt = dBt σ St + dt μ St + $\frac{(dBt)^2}{2}\sigma^2$St
dSt = dBt σ St + dt μ St + $\frac{dt}{2}\sigma^2$St
$\frac{dSt}{St}$ = dBt σ +dt(μ+$\frac{\sigma^2}{2}$)
Hence, $\frac{dSt}{St} = xdBt + ydt$
Where x = σ and y is (μ+$\frac{\sigma^2}{2}$)

(ii) E[St] = E[So $e^{\mu t + \sigma Bt}$] = So $e^{\mu t}$ E[$e^{\sigma Bt}$]
Bt follows N(0,t)
By mgf
E[$e^{\sigma Bt}$] = $e^{\frac{\sigma^2 t}{2}}$
E[St] = So $e^{\mu t + \frac{\sigma^2 t}{2}}$

V[St] = E[S^2t] – (E[St])^2
V[St] = So^2 $e^{2\mu t}$ E[$e^{2\sigma Bt}$] – So^2 $e^{2\mu t + \sigma^2 t}$
By mgf,
E[$e^{2\sigma Bt}$] = $e^{\frac{4\sigma^2 t}{2}}$ = $e^{2\sigma^2 t}$

V[St] = So^2 $e^{2\mu t + 2\sigma^2 t}$ - So^2 $e^{2\mu t + \sigma^2 t}$ = $So^2 e^{2\mu t}$ $e^{(2\sigma^2 t - \sigma^2 t)}$

(iii) Cov[St1,St2] when 0<t1<t2


(iii) Cov(St1, St2) & E[St2|St1] when 0<t1<t2
Cov(St1,St2) = E[St1 St2] – E[St1][St2]
E[St1] = So $e^{\mu t_1 + \frac{1}{2}\sigma^2 t_1}$
E[St2] = So $e^{\mu t_2 + \frac{1}{2}\sigma^2 t_2}$
E[St1 St2] = E[So $e^{\mu t_1 + \sigma B t_1}$. So $e^{\mu t_2 + \sigma B t_2}$]
= So² $e^{\mu(t_1+t_2)}$ E[$e^{\sigma B t_1 + \sigma B t_2}$]
= So² $e^{\mu(t_1+t_2)}$ E[$e^{\sigma(B_{t_1}+B_{t_2})}$]
= So² $e^{\mu(t_1+t_2)}$ E[$e^{\sigma(B_{t_1}+(B_{t_1}+B_{t_2}-B_{t_1}))}$]
E[exp[σ(2Bt1 + (Bt2 − Bt1))]]
E[exp[2σBt1]] E[exp[σ(Bt2 − Bt1)]] → By mgf.
E[St1 St2] = So² $e^{\mu(t_1+t_2)}$ $e^{0+\frac{1}{2}(4\sigma^2 t_1)}$ $e^{0+\frac{1}{2}(\sigma^2(t_2-t_1))}$
= So² $e^{\mu(t_1+t_2)}$ $e^{2\sigma^2 t_1}$ $e^{\frac{1}{2}\sigma^2(t_2-t_1)}$
Cov(St1, St2) = So² $e^{\mu(t_1+t_2)}$ $e^{2\sigma^2 t_1}$ $e^{\frac{1}{2}\sigma^2(t_2-t_1)}$ − So $e^{\mu t_1 + \frac{1}{2}\sigma^2 t_1}$ So $e^{\mu t_2 + \frac{1}{2}\sigma^2 t_2}$
= So² $e^{\mu(t_1+t_2)}$ [$e^{2\sigma^2 t_1 + \frac{1}{2}\sigma^2 t_2 - \frac{1}{2}\sigma^2 t_1}$ − $e^{\frac{1}{2}\sigma^2 t_1 + \frac{1}{2}\sigma^2 t_2}$]
= So² $e^{\mu(t_1+t_2)}$ ($e^{\frac{3}{2}\sigma^2 t_1}$ − $e^{\frac{1}{2}\sigma^2 t_1}$) $e^{\frac{1}{2}\sigma^2 t_2}$

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7. dC = μCdt + σCdWt
recombining binomial tree
(i) first step of binomial process

	p	S0*u
S0	Probabilities	
	(1-p)	S0*d

Doubt

8. binomial
(i) Derive the Price of option in one step binomial model using risk neutral valuations.
The share price S0 can either move upwards to S0u or downwards to S0d, according to one step binomial model. and d < e^r< u as principal of no arbitrage applies.
The derivative pays cu if the price goes up and cd if the price goes down.
So, in a portfolio, at time 0,
There are φ units of cash and $\emptyset$ units of stock
$\mathrm{V} = \emptyset\ \mathrm{So} + \varphi$
At time 1,
It's either ;
$\mathrm{V1} = \emptyset\ \mathrm{Sou} + \varphi\mathrm{e\hat{}r}$ or $\emptyset\ \mathrm{Sd} + \varphi\mathrm{e\hat{}r}$
Therefore,
$\mathrm{Cu} = \emptyset\ \mathrm{Sou} + \varphi\mathrm{e\hat{}r}$ and $\mathrm{Cd} = \emptyset\ \mathrm{Sd} + \varphi\mathrm{e\hat{}r}$
Solving two equations simultaneously, we get
$\emptyset\ = \frac{Cu - Cd}{S0(u-d)}$ and $\varphi = \frac{e^{-r}[uCd - dCu]}{u-d}$
$inputing\ the\ values\ back\ in\ equation\ at\ time\ 0$:
V0 = $e^{-r}(Cuq + (1-q)Cd)$
Where q = $\frac{e^r - d}{u-d}$ and q is the risk neutral probability.

(ii) Price of the option does not incorporate the probabilities of the stock going up or stock going down even though it is intuitive to assume. As we observe the formula, the price of the option is calculated on the potential derivative payoffs of upward and downward movement with risk neutral probability introduced by us, which also just depends on u, d and r.

(iii) Show that underlying stock price grows on average at risk free rate
Expected value of stock price at time 1 can be written as
E(St1) = S0 (qu + (1-q)d)
And substituting the values of q and (1-q)
We get E(St1) = S0 e^r
Hence we can see that the underlying stock price grows at risk free rate on average.

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(iv) In terms of risk neutral world,
People do not require compensation for risk as they are indifferent to risk. Hence, expected return on all securities and options is the risk free intere

9. Table

(i) Calculate price of 3 month forward on iota ltd
F = Soe^{rT}
Where F is the price of forward, So is the stock price, r is the risk-free rate continuously compounded and t is the time to expiry.
Put call parity says that:
c + Ke^{-rt} = p + So
since we need So and r, we can input 2 values from the table and solve simultaneously to get So and r
13.334 + 70$e^{-r*3/12}$ = 0.120 + S0
8.869 + 75 $e^{-r*3/12}$ = 0.568 + S0
Solving simultaneously, we get
r = 7.02%
S0 = 81.996
Hence,
Price of forward is 83.447

(ii) Calculate forward rate for delivery between 3 months from now and 6 months from now

(iii) According to put call parity,
c + Ke^{-rt} = p + So
To find a,
6.889 + a*$e^{-0.0702*0.25}$ = 1.055 + 81.996
a = 77.51

To find b,
b + 80*$e^{-0.0702*0.25}$ = 1.789 + 81.996
b = 5.1768

To find c,
2.594 + 85*$e^{-0.0702*0.25}$ = c + 81.996
c = 4.119

10. N step recombining binomial model
Each time period is one month
Interest rates are assumed to be zero

(i) Show that the risk neutral step up probability q is less than 0.5

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q = $\frac{e^r - d}{u - d}$ $and\ since\ interest\ rates\ are\ 0$,

q = $\frac{1 - d}{u - d}$

in recombining model d = $\frac{1}{u}$

substituting this value, we get

q = $\frac{1}{u+1}$

and if no arbitrage principle is supposed to hold, d < 1 < u

then, u > 1

u+1 > 2

$\frac{1}{u+1}$ < ½

$q < \frac{1}{2}$

hence proved.

11. Sum

(i) Recombining binomial tree: here, the size of upward movement (u) and downward movement (d) are assumed to be the same under all states and all time intervals. And hence it also has a constant risk neutral probability q at all times and states.
Main advantage is that it only has n+1 possible states of time as opposed to 2^n states in non-recombining binomial tree. This reduces the model by a large extent.
Disadvantage is that it assumes the upward and downward movement to be constant which may not be true in real world scenario.

(ii) Two step binomial tree
6 month European put option
K = 950
S0 = 1000
For first three months,
U1 = 1.1
D1 = 0.95
Rf1 = 1.75%
For next three months,
U2 = 1.2
D2 = 0.9
Rf2=2.5%

(a) Value of put option

q1 = $\frac{e^r - d}{u - d}$ = 0.4510

q2 = 0.41772

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K=950

State	St	payoff
V1(1) = So u	1100	0
V1(2) = So d	950	0
V2 (1)= So u u	1320	0
V2(2) = So u d	990	0
V2(3) = So d u	1140	0
V2 (4)= So d d	855	95

V1(1) = $e^{-r}\ (Cu\,q\ +\ (1-q)\ Cd)$
V1(1) = 0
V1(2) = $e^{-r}\ (Cu\,q\ +\ (1-q)\ Cd)$
V1(2) = $e^{-0.025}(0.41772\ *\ 0\ +\ 0.58228\ *95)$
V1(2) = 53.9508

V0 = $e^{-r}\ (V1(1)q\ +\ V1(2)(1-q))$
V0 = $e^{-0.0175}(0.4510*0+53.9508*0.5490)$
V0 = 29.105

(b) Main disadvantage would be to have appropriate values for each time step , i.e. values for u, d and r at each step. This would produce more accurate answer but it is difficult to obtain all these parameters.
Also, a new tree would have 2^6 i.e. 64 nodes and this is extensive.
An alternative model is 6 step recombining tree that will only have 6+1 i.e. 7 nodes and constant values of u and d.

12. Zt is the standard Brownian motion, derive SDEs and explain which process has zero drift

a. U(t) = 2Zt – 2
By Ito's Lemma,
dUt = dZt 2 – 0
dU(t) = 0dt + 2dZt

U(t) has zero drift.

b. V(t) = (Zt)^2 – t
By Ito's Lemma,
dV(t) = dZt (2*Zt) + dt(-1) + $\frac{(dZt)^2}{2}*2$
dV(t) = 2Zt dZt -dt + dt
dV(t) = 2Zt dZt

V(t) has zero drift.

c. W(t) = $t^2 Z(t)$ - 2$\int_0^t s\ Z(s)\ ds$

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By Ito's Lemma,
dW(t) = dZt (—
doubt

13. At the money American put option
S0 = 200
Rf = 10% pa
σ = 35% pa
t=2 months
(i) Values of u and d
St = S0 $e^{((\mu-1/2\ \sigma^{\wedge}2\ )T\ +\ \sigma dWt)}$, since stock price follows geometric Brownian motion
Thus, $\frac{St}{So}$ ~ ln N((μ - $\frac{\sigma^2}{2}$)T, σ^2T)
Assumption: u = $\frac{1}{d}$
$e^{(\mu+1/2\sigma^2)}$ = mean of a log normal distribution
E($ST/S0$) = $e^{\left((\mu-1/2\sigma^2)dt+1/2\sigma^2 dt\right)}$= $e^{\mu}dt$
E(ST)=S0 $e^{\mu}dt$
The variance is given by:
Var(ST/S0)=E (ST/S0)^2-(E(ST/S0))^2
$= lnu^2q + (-lnu)^2(1-q) - (S0e^{\mu}dt)\sigma^2 dt$
$= lnu^2q + (-lnu)^2(1-q) - (S0e^{\mu}dt)\sigma^2 dt$
$= (-lnu)^2\sigma\sqrt{d}t$
$= lnu$
Thus, $u = e^{(\sigma\sqrt{d}t)}andd = e^{(-\sigma\sqrt{d}t)}$

(ii) Value of American put option using 2 time step
q=0.5161
Payoff at time 1= 0, 9.85 and payoff at time 2= 0, 0, 6.62, 19.22
Value of (holding the) option a time 1=$e^{(-0.1*1/12)}\left(6.62 \times q + 19.22(1-q)\right)$
=12.61. This is more than the payoff of exercising the option (9.85). Thus, we will hold the option.
Value at time 0= $e^{(-0.1*1/12)}(0 + (1-q)12.61)$
Value of put option at time 0=6.05

14. S0 = 500
One-time step : 3 month period
T = 2
u = 1.06
d = 0.95
rf = 5% pa continuously compounded

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(i) Value of six-month European call option with K = 510

Payoff diagram for European call:

State	St	payoff
V1(1) = So u	530	20
V1(2) = So d	475	0
V2 (1)= So u u	561.8	51.8
V2(2) = So u d	503.5	0
V2(3) = So d u	451.25	0

q = $\frac{e^r - d}{u - d}$ = 0.5689

Value of 6 month European call= $e^{-r}(V1(1) * q + V1(2) * (1 - q))$

Where V1(1) = $e^{-r}(V2(1) * q + V2(2) * (1 - q))$

V1(2) = $e^{-r}(V2(2) * q + V2(3) * (1 - q))$

Hence, V0 = 16.351

(ii) Payoff diagram for European put:

State	St	payoff
V1(1) = So u	530	0
V1(2) = So d	475	35
V2 (1)= So u u	561.8	0
V2(2) = So u d	503.5	6.50
V2(3) = So d u	451.25	58.75

Value of 6 month European put = $e^{-r}(V1(1) * q + V1(2) * (1 - q))$

Where V1(1) = $e^{-r}(V2(1) * q + V2(2) * (1 - q))$

V1(2) = $e^{-r}(V2(2) * q + V2(3) * (1 - q))$

Hence, V0 = 13.759

Using put call parity,

C + Ke^-rt = P + S0

LHS: 16.351 + 510$e^{-0.05*1/2}$ = 513.759

RHS: 13.759 + 500 = 513.758

Hence proved.

(iii) For american option , payoffs are same, although only difference is the time when the option can be exercised. So we calculate the value of put at time 1 too, to check if it is advisable to exercise early.

V1(1) = $e^{-0.05*1/4}(6.5 * 0.4311 \ + \ 0)$ = 2.767

V1(2) = $e^{-0.05*1/4}(6.5 * 0.5689 \ + \ 58.75 \ * 0.4311)$ = 28.65

However, payoff by immediate exercise at V1(2) is 35.

35>28.65

Advisable to exercise immediately at this point.

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Value of American put is
$e^{-0.05*1/4}(35 * 0.4311\ +\ 2.767 * 0.5689)$ = 16.46

(iv) Expected return = 9%
So expected payoff in 3 months is calculated using this real rate of return.
Thus p = $\frac{e^{0.09*3/12}-0.95}{1.06\ -\ 0.95}$
p=0.6614
The payoff at time V1(1) of the call option was 20
Hence the expected payoff = 20*0.6614= 13.228.
However, we might not know the correct discount rate that is to be used to compute the real world probability and hence the expected payoff. Therefore, it is easier to use risk neutral valuations wherein the asset is expected to earn only the risk-free rate.

15. dS = μSdt + σSdz
(i) process followed by S^k
f(St) = S^k
$\frac{\partial F}{\partial St}$ = k*S^{k-1}
$\frac{\partial^2 F}{\partial St^2}$ = k(k-1)* $S^{\,k-2}$

$By\ ito's\ Lemma,$
$dF = dSt\ * k\,S^{k-1} + (dSt)^2/2\ *\ \mathrm{k(k-1)} *\ S^{\,k-2}$
df = (μSdt + σSdZt) $k\,S^{k-1}$ + $(\sigma^2 S^2 dt)$*$\frac{k(k-1)}{2}$ $S^{\,k-2}$
df = μk f dt + σk f dZt + $\sigma^2\,\frac{k(k-1)}{2}$ f dt
df = (μk + $\sigma^2\,\frac{k(k-1)}{2}$) f dt + σk f dZt

Hence,

S^k follow geometric Brownian motion with drift i.e. μ1 = μk + $\sigma^2\,\frac{k(k-1)}{2}$ and volatility σ1 = σk.

f/fo follows lognormal distribution with mean (μ1-1/2 σ1^2) and variance with σ1^2 t.

(ii) f(St,t) = St e^{-rt}

$\frac{\partial F}{\partial St}$ = e^{-rt}

$\frac{\partial^2 F}{\partial St^2}$ = 0

$\frac{\partial F}{\partial t}$ = -re^{-rt}St

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By ito's lemma,
df = -r$e^{-rt}S$t dt + dSt e^{-rt}
substituting dSt,
df = (μ-r)f dt + σf dZt
process is martingale is and only if the drift term i.e. dt term is 0.
Here, if μ = r, then the process is a martingale.

(iii) For S^k $to\ be\ a\ martingale,$
We need drift term to be 0 i.e. μk + $\sigma^2 \frac{k(k-1)}{2}$ to be 0.
If the values solve to be 0, we can say that the process is a martingale.