Name: Darshan Lodha
Roll No: 433
Semester 4
Assignment Number 1

Statistical And Risk Modelling 2

Q 1] i)

Ans a A stochastic model refers to the type of model wherein the input variables follow a time dependent random phenomenon.

b. The advantages of a stochastic model over a deterministic model are as follows:

i. A stochastic model allows the randomness of input variables to be incorporated in the model which is more practical relative to a deterministic model which incorporat[es] a deterministic i.e. a constant value at a particular point of time.

ii Stochastic model allows the use of advanced numerical methodologies like Monte Carlo simulation, which is an extremely powerful technique for solving complex problems

(ii)

Ans a For a time inhomogeneous model, the transition rates are a function of time t.

b. It is certainly possible to improve the fit by using a time inhomogeneous model in this instance. However, if the age profile is represented by a density function $f(a)$, then the overall average rate at which a healthy employee falls sick is $= \int f(a)\sigma(a)\,da$, roughly constant for all t.

c The same of course applies to the overall average rate of recovery.

(iii) Required proof:
Given that the non-overlapping increments of the stochastic process are independent, $X_{t+u} - X_t$ for every $u > 0$ are independent of past values X_m and are non-overlapping.
Thus,

$$Y = P[X_t = a \mid X_{s_1} = x_1, X_{s_2} = x_2, \ldots, X_{s_n} = x_n, X_s = x]$$

Because, $X_s = x \longrightarrow X_t = X_t - X_s + x$

$$\therefore Y = P[X_t - X_s + x = a \mid X_{s_1} = x_1, X_{s_2} = x_2, \ldots, X_s = x]$$

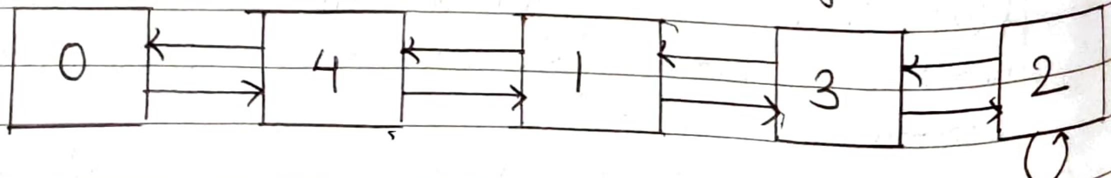
$\because X_t - X_s$ are independent of past values of X, therefore $X_t - X_s + x$ will also be independent of past values of X_m.

$$\therefore Y = P[X_t - X_s + x = a \mid X_s = x]$$

$$\therefore Y = P[X_t = a \mid X_s = x] \qquad (\because X_s = x)$$

Thus, independent increments satisfy Markov property.

Q.2] (i).
Ans. Consider a Markov chain taking values $S = \{0, 1, 2, 3, 4\}$ such that it represents the number of umbrellas in the place where the actuary currently is (home or office)



(ii)
Ans:

$$P = \begin{array}{c c} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & q & p \\ 0 & 0 & q & p & 0 \\ 0 & q & p & 0 & 0 \\ q & p & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(iii)
Ans: Using the condition of stationarity,

$$\Pi \cdot P = \Pi$$

$$\therefore (\Pi_0 \ \Pi_1 \ \Pi_2 \ \Pi_3 \ \Pi_4) \cdot P = (\Pi_0 \ \Pi_1 \ \Pi_2 \ \Pi_3 \ \Pi_4)$$

$$\therefore \Pi_0 = \Pi_4 \cdot q \quad \text{—— (1)}$$

$$\therefore \Pi_1 = \Pi_3 q + \Pi_4 p \quad \text{—— (2)}$$

$$\therefore \Pi_2 = \Pi_2 q + \Pi_3 p \quad \text{—— (3)}$$

$$\therefore \Pi_3 = \Pi_1 q + \Pi_2 p \quad \text{—— (4)}$$

$$\therefore \Pi_4 = \Pi_0 + \Pi_1 p \quad \text{—— (5)}$$

$$\therefore \Pi_0 + \Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 = 1 \quad \text{—— (6)}$$

Solving above equations, we get

$$\Pi_0 = \frac{q}{4+q} \text{ and } \Pi_1 = \Pi_2 = \Pi_3 = \Pi_4 = \frac{1}{4+q}$$

(iv)

Ans. He will get wet every time he happens to be in state 0 and it rains.

The chance he is in stage zero is π_0. The chance that it rains is p. Hence, probability that he gets wet is,

$$P(\text{getting wet}) = P(\text{being in state 0 and it rains})$$

$$= \left(\frac{q}{4+q}\right)p$$

(v)

Ans. Given : $p = 0.6$

$$P(\text{getting wet}) = 0.0545$$

If he wants the chance to be less than 1%, then clearly he needs more umbrella. Let's suppose he needs N umbrellas

$$\pi_1 = \pi_2 = \pi_3 = \cdots = \pi_N = \frac{\pi_0}{q}$$

But, $\pi_0 + \pi_1 + \pi_2 + \cdots + \pi_N = 1$

Thus, $\pi_N \cdot q + N \cdot \pi_N = 1$

$$\therefore \pi_N = \frac{1}{N+q}$$

$$P(\text{getting wet}) = p \cdot \pi_0 = \frac{pq}{N+q}$$

For, P(getting wet) < 1/100

$$\frac{p+q}{N+q} < \frac{1}{100}$$

$$N > 100 \cdot p \cdot q - q$$

$$\therefore N > 23.6$$

Thus it is not worth to keep 24 umbrellas instead of 4 to reduce the probability of getting wet from 67 to 1%.

Q.3] (i) and (ii).

Ans

	Process	State Space	Time set
1	Counting process	Discrete	Either
2	General random walk	Either	Discrete
3	Poisson process	Discrete	Continuous
4	Markov chain	Discrete	Discrete
5	Markov jump process	Discrete	Continuous.

Q.4] (i)

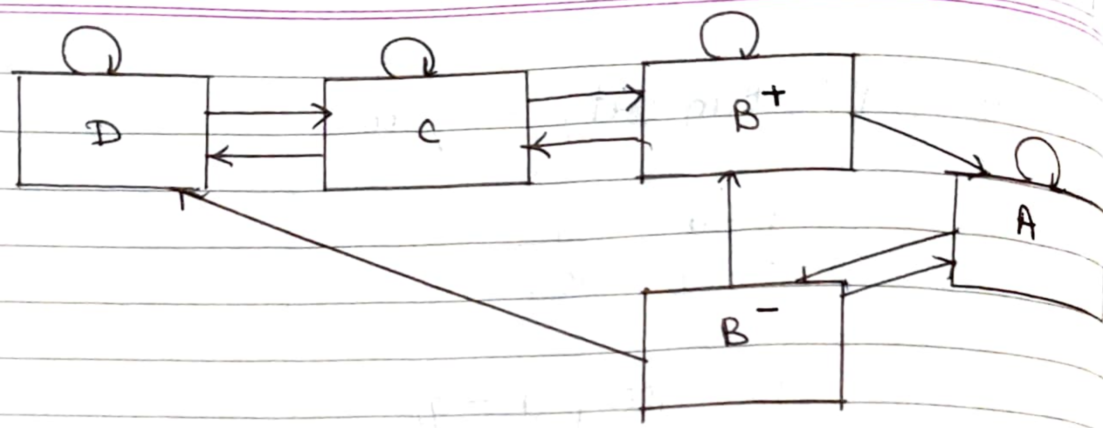
Ans (a) Past history of the sportsmen is required to decide where to go further in the Markov chain.

(b) If a sportsman is at A and his performance reduces, we need to know what level of performance he was at the previous year to determine whether he or she drops one or two levels in the chain

(ii)

Ans. Let B^+ be the level with no reduction in performance parameter last year and B^- be the level with reduction in performance parameter last year.

(iii)
Ans



(iv)
Ans

$$P = \begin{array}{c} \\ A \\ B^+ \\ B^- \\ C \\ D \end{array} \begin{array}{c} \begin{array}{ccccc} A & B^+ & B^- & C & D \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.3 & 0.2 & 0.5 & 0 & 0 \\ 0 & 0.3 & 0.2 & 0 & 0.5 \\ 0.3 & 0 & 0.2 & 0 & 0.5 \\ 0 & 0 & 0 & 0.3 & 0.7 \end{bmatrix} \end{array}$$

(v)
Ans: Using condition of stationarity.

$$\pi = \pi \cdot P$$

We get by solving,

$$\pi_A = 0.42445$$
$$\pi_{B^-} = 0.12733$$
$$\pi_{B^+} = 0.12733$$
$$\pi_C = 0.15280$$
$$\pi_D = 0.16808$$

(vi)
Ans The long run average contract value is,

$$100\%\,\pi_D + 120\%\,\pi_C + 150\%\,(\pi_{B^+} + \pi_{B^-}) + 175\%\,\pi_A$$

Long run average contract value = £1.4762 million

(vii)

Ans: Let 'm' be the number of transitions (and years) taken to reach level A from any current level 'i'. Then,

$$m_D = 1 + 0.5m_D + 0.5m_C \quad \text{— ①}$$

$$m_C = 1 + 0.2m_C + 0.5m_B + 0.3m_D \quad \text{— ②}$$

$$m_{B^+} = 1 + 0.2m_{B^+} + 0.5m_A + 0.3m_C \quad \text{— ③}$$

$$m_{B^-} = 1 + 0.3m_D + 0.5m_A + 0.2m_{B^+} \quad \text{— ④}$$

$$m_A = 0 \quad \text{— ⑤}$$

Solving, we get,

$$m_C = 7.12 \text{ years}$$

and $m_D = 2 + m_C = 9.12$ years.

Q.5] (i)

Ans (a) There is an explicit dependence on the past behaviour of $\{Y_j : j \leq n\}$, in the probability distribution of Y_{n+1}.

(b) Further, X_n is nothing but the sum of Y_j. Hence, the markov property does not hold.

(ii)

Ans a: In the above subquestion, we show that there is an explicit dependence on the past behaviour of $\{Y_j ; j \leq n\}$ and hence, Markov property does not hold.

b This implies that the sequence $\{Y_n : n \geq 1\}$ does not form a markov chain.

(iii)

Ans.

$$P = \begin{bmatrix} p & 1-p & 0 & 0 & 0 & 0 \\ 0 & pe^{-\lambda} & 1-pe^{-\lambda} & 0 & 0 & 0 \\ 0 & 0 & pe^{-2\lambda} & 1-pe^{-2\lambda} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(iv)

Ans. (a) The chain is time homogeneous since the transition probabil[ity] calculated is independent of time 'n'.

(b) It is irreducible

(c) There can be no stationary distribution

(v)

Ans. $P(\text{No further error}) = (pe^{-j\lambda})^n = p^n e^{-nj\lambda}$

Q.6] (a)

Ans. 1 A stochastic model allows for the randomness of input parameters.

2 (Advantages same as those mentioned in Q.1)

(b) The following subquestions are not there in syllabus as part of proofs.

Q.7] (a)

Ans. 1 The process maybe expressed as a Markov chain by considering the following states. Each state indicates the current number of successive defeats.

2. Let, 1 = 0 → Not defeated last time

2 = 1 → Defeated in the last match

3 = 2 → Defeated in the last 2 matches

4 = 3 → Defeated in the last 3 matches

5 = 4 → Captain sacked.

$$P = \begin{bmatrix} 0.7 & 0.3 & 0 & 0 & 0 \\ 0.7 & 0 & 0.3 & 0 & 0 \\ 0.7 & 0 & 0 & 0.3 & 0 \\ 0.7 & 0 & 0 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b)

Ans (1) A Markov chain is said to be irreducible if any state j can be reached from any state i.

(2) This process is not irreducible.

(c)

Ans 1 P (Remaining captain for exactly 4 matches) = 0.0081

2 P (Remaining captain for exactly 5 matches) = 0.00567

3 P (Remaining captain for exactly 7 matches) = 0.00567

4 P (Remaining captain for exactly 9 matches) = 0.0056241

(d)

Ans: Using the stationary distribution condition,

$N_1 = 174.94$

$N_2 = 171.60$

$N_3 = 160.49$

$N_4 = 123.46$

Thus, $E[N] = 174.94$ matches.

Q.8] (i)

Ans (a) Markov property holds.

(b) The state space is defined as follows:

T = Tie

L_F = Federer leads.

L_N = Nadal leads
G_F = Federer wins
G_N = Nadal wins

(ii)
Ans.

$$P = \begin{bmatrix} 0 & 0.55 & 0.45 & 0 & 0 \\ 0.45 & 0 & 0 & 0.55 & 0 \\ 0.55 & 0 & 0 & 0 & 0.45 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(iii)
Ans. (a) The chain is irreducible
(b) The chain is aperiodic

(iv)
Ans. P(Returning to tie after 2 points) $= 0.55 \times 0.45 + 0.45 \times 0.55$
$= 0.495$

We need to find the number of such cycles of returning to tie such that,

$$0.495^N = 1 - 0.95$$
$$\therefore N = 4.26$$

Required number of cycles = 5.

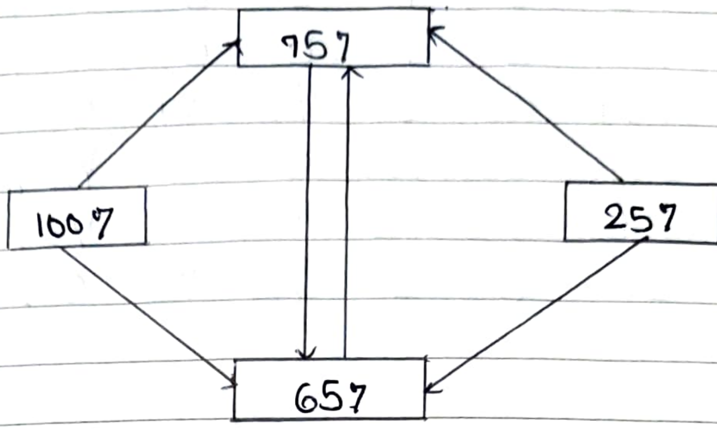
(v)
Ans. Let F_T be the probability that Federer wins and let N_T be the probability that Nadal wins

$$\therefore N_T = 0.2025 + 0.495 \times N_T$$
$$\therefore N_T = 0.401$$

Thus, $F_T = 0.599$

Q.9] (a)

Ans:



(b)

Ans: Using the condition for stationary distribution,
We get,

$x_1 = 0$

$x_2 = 0.5423$

$x_3 = 0.3559$

$x_4 = 0.1010\%$

(c)

Ans: The steady state proportion of customers having a 65% target allocation for xyz ltd is 35.59%.
It is noted that initially this proportion is 45% and is quickly dropping to:
37.5% in one year's time and 36.2% in two year's time

Q.10]

Ans: This question is very similar to Question 2. Thus, it has a very similar solution.

Q.11] (i)

Ans: If each room is represented by a state, then the transition matrix P, is given as:

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

(ii)
Ans. a The chain is irreducible
b. The chain is not aperiodic

(iii)
Ans. For the process to be stationary,

$$\pi = \pi P.$$

We need to satisfy above condition.

(iv)
Ans. We find that the mean recurrence time for room 1 is

$$\frac{1}{\pi(1)} = 12$$

(v)
Ans. Expected time to return to room 1 after calculating turns out to be 9.

Q.12] (i)
Ans. We need to find $m_{2,4}$.
Now, $m_{i,4} = 0$ if $i=4$ and $1 + \sum p_{ij} m_{j,4}$ if $n \neq t$

$\therefore \quad m_{2,4} = 1 + (1 \cdot m_{3,4})$

$m_{4,4} = 0$

$\therefore m_{3,4} = 1 + \left(\frac{1}{3} m_{4,4}\right) + \left(\frac{2}{3} \times m_{2,4}\right)$

$\therefore m_{3,4} = 1 + 0 + \frac{2}{3}(1 + m_{3,4})$

$\therefore m_{3,4} = 5$

Hence,

$m_{2,4} = 1 + 5 = 6$ steps.

(ii)
Ans. $P(1,2,3,2,3,4) = P(X_0 = 1) \times p_{12} \times p_{23} \times p_{32} \times p_{23} \times p_{34}$

$= \frac{3}{4} \times \frac{3}{5} \times 1 \times \frac{2}{3} \times 1 \times \frac{1}{3}$

$P(1,2,3,2,3,4) = \frac{1}{10}$

Q.13] (i) (a)
Ans. From the data given, we can say,

$P_{RR} = 6/14 = 3/7$
$P_{RS} = 8/14 = 4/7$
$P_{SR} = 8/15$
$P_{SS} = 7/15$

(b)
Ans. P(Rain on 3rd July 2020) = 0.142222 + 0.130612 + 0.130612 + 0.078717

$= 0.482164$

(ii) (a)

Ans: $\mu = 3$

$\therefore P(x>0) = 1 - P(x=0)$

$\because$ It follows Poisson distribution,

$\therefore P(x>0) = 0.95021$

(b)

Ans: $P(2 \leq x < 5) = P(x=2) + P(x=3) + P(x=4)$

$= 0.61611$

(c)

Ans: Average number of policies sold per day $= 3/5 = 0.6$

So, on a given day, $P(x) = \cancel{0.34}\ 0.32929$.