SRM Assignment 2

Sabyasachi Rathore
Roll No - 434

1. (i) $\begin{array}{cc} B \\ O \end{array} \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$

(ii) The holding times are exponentially distributed with parameter λ in state B and μ in state O.

(iii) $\frac{d}{dt} {}_tP_s^{BB} = -\lambda \cdot {}_tP_s^{BB} + \mu \times {}_tP_s^{BO}$

$\frac{d}{dt} {}_tP_s^{BO} = \lambda \cdot {}_tP_s^{BB} - \mu \cdot {}_tP_s^{BO}$

(iv) We have a 2 state model so -
${}_tP_s^{BB} + {}_tP_s^{BO} = 1$

Substituting -

$\frac{d}{dt} {}_tP_s^{BB} = -\lambda \times {}_tP_s^{BB} + \mu(1 - {}_tP_s^{BB})$;

$\frac{d}{dt}\left[\exp((\lambda+\mu)t) \times {}_tP_s^{BB}\right] = \mu \times \exp((\lambda+\mu)t)$

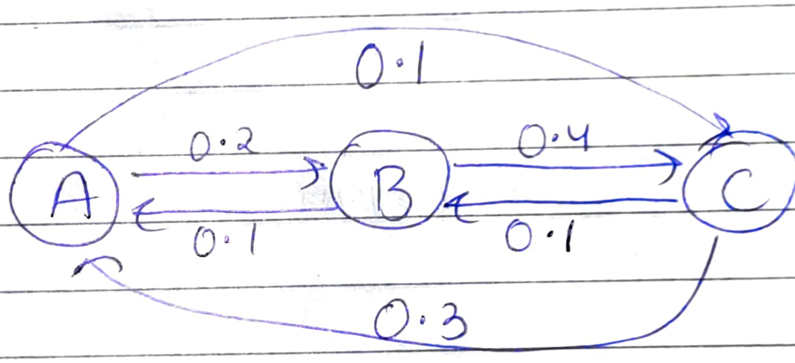
Hence,

$$\exp((\lambda+\mu)t) \times {}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} \times \exp((\lambda+\mu)t) + \text{constant}$$

Since the process is in state Bid at time s the constant is $\frac{\lambda}{\lambda+\mu}$

thus, $${}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \times \exp(-(\lambda+\mu)t)$$

2. (i)



(ii) $$\frac{d}{dt} P_{AA}(t) = -0.3\, P_{AA}(t) + 0.1\, P_{AB}(t) + 0.3\, P_{AC}(t)$$

$$\frac{d}{dt} P_{AB}(t) = 0.2\, P_{AA}(t) - 0.5\, P_{AB}(t) + 0.1\, P_{AC}(t)$$

$$\frac{d}{dt} P_{AC}(t) = 0.1\, P_{AA}(t) + 0.4\, P_{AB}(t) - 0.4\, P_{AC}(t)$$

(iii) Case I – Stay in state 1

$$\frac{d}{dt} P_{\bar{A}\bar{A}}(t) = -0.3\, P_{\bar{A}\bar{A}}(t)$$

$$P_{\bar{A}\bar{A}}(t) = \exp(-0.3t)$$

for $t = 2$, $\exp(-0.6) = 0.5488$

(iv) For third jump into state C →

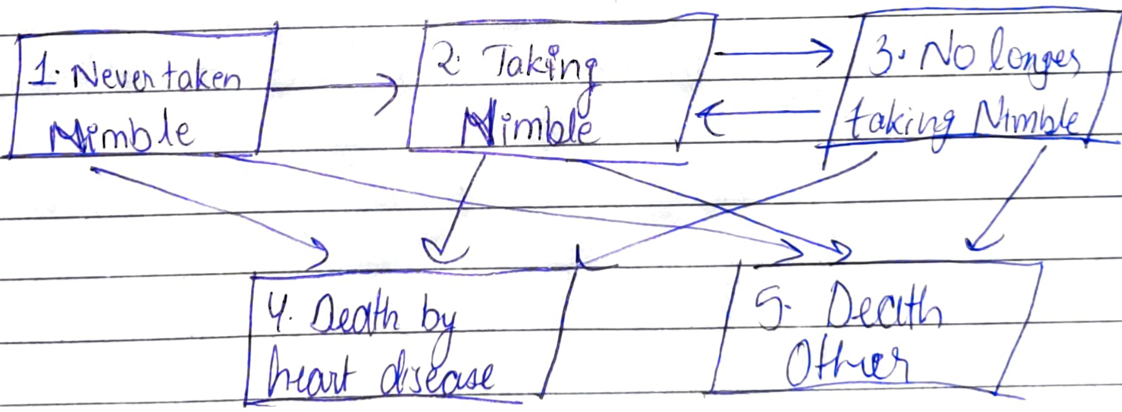
$$BAC = \frac{2}{3} \times \frac{1}{5} \times \frac{1}{3} = \frac{2}{45}$$

$$CAC = \frac{1}{3} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{12}$$

$$CBC = \frac{1}{3} \times \frac{1}{4} \times \frac{4}{5} = \frac{1}{15}$$

$$\text{Sum} = \frac{7}{36} \approx 0.194$$

3.(i)



(ii) Chapman Kolmogrov equation →

$${}_{dt+t}P_x^{34} = {}_tP_x^{31}\,{}_{dt}P_{x+t}^{14} + {}_tP_x^{32}\,{}_{dt}P_{x+t}^{34} + \dots$$

Since ${}_{dt}P_{x+t}^{54} = {}_tP_x^{31} = 0$

$${}_{dt+t}P_x^{34} = {}_tP_x^{32}\,{}_{dt}P_{x+t}^{24} + {}_tP_x^{33}\,{}_{dt}P_{x+t}^{34} + {}_tP_x^{34}\,{}_{dt}P_{x+t}^{44}$$

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Given that ${}_{dt}P_{x+t}^{44} = 1$

And assuming that, for small dt

$${}_{dt}P_{x+t}^{ij} = \mu_{x+t}^{ij}\,dt + o(dt) \qquad [i \neq j]$$

where $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$

then substituting, we have

$${}_{dt+t}P_x^{34} = {}_tP_x^{32}\,\mu_{x+t}^{24}\,dt + {}_tP_x^{33}\,\mu_{x+t}^{34}\,dt + {}_tP_x^{34} + o(dt)$$

so that, ${}_{dt+t}P_x^{34} - {}_tP_x^{34} = {}_tP_x^{32}\,\mu_{x+t}^{24}\,dt + {}_tP_x^{33}\,\mu_{x+t}^{34}\,dt + o(dt)$

4. (i) 3 calls per hour.

(ii) Expected time for next call is 20 minutes.

(iii) Prob. of 0 calls in 0.5 hour

Using $P_j(t) = e^{-\lambda t}(\lambda t)^j / j!$

$$\Rightarrow P_0(0.5) = \frac{e^{-1.5}(1.5)^0}{0!}$$

$$P_0(0.5) = e^{-1.5} = 0.2231$$

(iv) calls x duration = 3 x 7 = 21 minuts

$$\text{Prob} = \frac{21}{60} = \underline{\underline{0.35}}$$

5. Chapman Kolmogrov Equation is

$$P_{HH}(x, t+dt) = P_{HH}(x,t) P_{HH}(t, t+dt) + P_{HS}(x,t) P_{SH}(t, t+dt) + P_{HD}(x,t) P_{DH}(t, t+dt)$$

So:

$$P_{HH}(x, t+dt) = P_{HH}(x,t) P_{HH}(t, t+dt) + P_{HS}(x,t) P_{SH}(t, t+dt)$$

Assuming that, for small dt

$$P_{ij}(t, t+dt) = \lambda_{ij}(t)\, dt + o(dt) \qquad [i \neq j]$$

$$P_{ii}(t, t+dt) = 1 + \lambda_{ii}(t) dt + o(dt)$$

OR

$$P_{ii}(t, t+dt) = 1 - \sum_{j \neq i} \lambda_{ij}(t)\, dt + o(dt)$$

where the λs are instantaneous rates & $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$

$$\therefore P_{HH}(x, t+dt) - P_{HH}(x,t) = P_{HH}(x,t)\left(-\sigma(t) - \mu(t)\right) dt + P_{HS}(x,t) \rho(t, C_t)\, dt + o(dt)$$

Hence, $$\frac{d}{dt} P_{HH}(x,t) = \lim_{dt \to 0} \frac{P_{HH}(x, t+dt) - P_{HH}(x,t)}{dt}$$

$$= P_{HH}(x,t)\left(-\sigma(t) - \mu(t)\right) + P_{HS}(x,t) \rho(t, C_t)$$

(ii) The equation simplifies when considering $P_{\overline{HH}}(t)$ to

$$\frac{d}{dt} P_{\overline{HH}}(0,t) = -\left(\sigma(t) + \mu(t)\right) P_{\overline{HH}}(t)$$

$$\frac{1}{P_{\bar{H}\bar{H}}(0,t)}\frac{d}{dt}P_{\bar{H}\bar{H}}(0,t) = -(\sigma(t)+\mu(t)) = \frac{d}{dt}\ln P_{\bar{H}\bar{H}}(t)$$

Integrate both sides:

$$\left[\ln P_{\bar{H}\bar{H}}(0,t)\right]_0^t = \int_{s=0}^{t} -(\sigma(s)+\mu(s))\,ds$$

as $P_{\bar{H}\bar{H}}(0) = 1$

$$P_{\bar{H}\bar{H}}(0,t) = \exp-\left(\int_{s=0}^{t}(\sigma(s)+\mu(s))\,ds\right)$$

6. All three processes have a discrete state space.

A markov chain and Markov jump chain both operate in discrete time but a Markov jump process operates in continous time.

All have the markov property.

7 (i) MLE of the transition intensity from state (i) to state 'j' is the number of transitions from state 'i' to state 'j' divided by the total waiting time in state i.

(ii) $P_{AA}(s,t)$
Using Integerate forward equations,

$$P_{AA}(s,t) = \exp\left(-\int_{u=s}^{t} \mu\,du\right)$$

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$$P_{AT}(s,t) = \int_{u=0}^{t} P_{AA}(s,u) \cdot \mu \cdot 1 du,$$

(iii) Measure from time zero ie $s=0$ & drop s from notation $\longrightarrow$

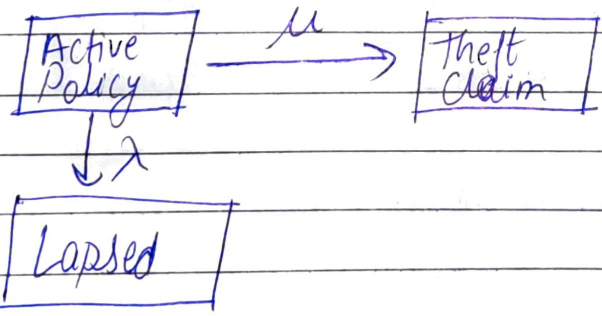
Solving the integrated forward equation

$$P_{AT}(T) = \int_{s=0}^{T} \exp(-\mu s)\mu ds = [-\exp(\mu s)]_0^T$$

$$= 1 - \exp(-\mu T)$$

Hence, expected cost is $C(1-\exp(-\mu T))$

(iv)



(v) We now have $\frac{d}{dt} P_{AA}(t) = -P_{AA}(t)(\mu+\lambda)$

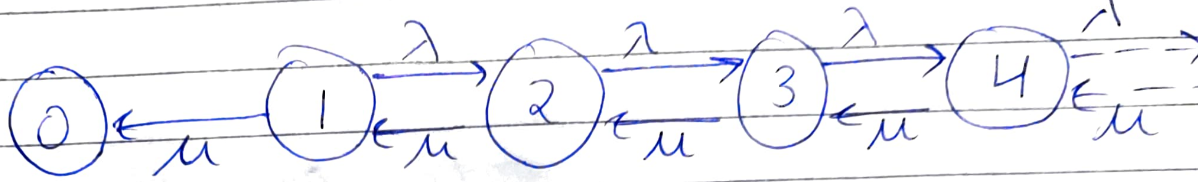
So $P_{AA}(t) = \exp(-(\mu+\lambda)t)$

We want $\frac{d}{dt} P_{AT}(t) = P_{AA}(t)\mu = \mu \exp(-(\mu+\lambda)t)$

So claims become $\frac{\mu}{\mu+\lambda} C(1-\exp(-(\mu+\lambda)T))$

8. (i) $\{0, 1, 2, 3, \ldots\}$

(ii)



(iii) Generator Matrix

$$\begin{array}{c|cccccc} \text{Lives} & 0 & 1 & 2 & 3 & 4 & \ldots \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \text{etc.} \\ \mu/(\mu+\lambda) & 0 & \lambda/\mu+\lambda & 0 & 0 & \\ 0 & \mu/(\mu+\lambda) & 0 & \lambda/\mu+\lambda & 0 & \\ 0 & 0 & \mu/\lambda+\mu & 0 & \lambda/\mu+\lambda & \\ 0 & 0 & 0 & \mu/\mu+\lambda & 0 & \\ \text{etc.} & & & & & \end{bmatrix}$$

(vi) $\left(\dfrac{\mu}{\mu+\lambda}\right)^3$

10 (i) $\dfrac{d}{dt} P_{\bar{A}\bar{A}}(t) = -2t \times P_{\bar{A}\bar{A}}(t)$

$\Rightarrow \dfrac{d}{dt}\left[\ln P_{\bar{A}\bar{A}}(t)\right] = -2t$

$\Rightarrow \ln P_{\bar{A}\bar{A}}(s) = -s^2 + \text{constant}$

We know $P_{\bar{A}\bar{A}}(0) = 1$ hence constant $= 0$

$\therefore P_{\bar{A}\bar{A}}(s) = \exp(-s^2)$

(ii) $\int_0^T P(\text{remains in } A \text{ to time } s) \times P(\text{transition to } B \text{ in time } s, s+ds) \times P(\text{remains in } B \text{ to time } T)\, ds$

$\Rightarrow \int_{s=0}^{T} P_{AA}(s) \times 2s \times P_{BB}(s,T)\, ds$

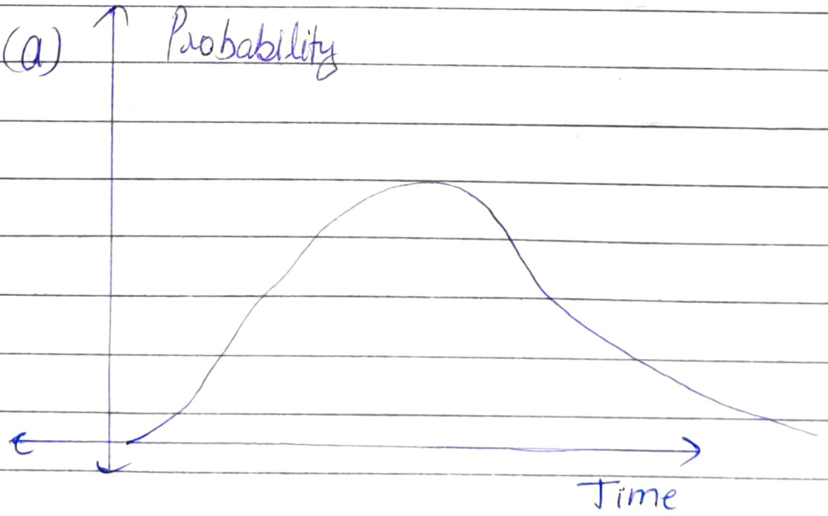
Using results from part (i), and similar result for P_{BB} with boundary condition $P_{BB}(s,s)=1$ gives us:

$$\int_{s=0}^{T} e^{-s^2} \times 2s \times e^{-T^2+s^2}\, ds$$

$$\Rightarrow \int_{s=0}^{T} 2s \times e^{-T^2}\, ds$$

$$\Rightarrow e^{-T^2} \times T^2$$

(iii) (a)



(b)

(c) $\frac{d}{dt}\left[e^{-t^2} \times t^2\right] = 2t \times e^{-t^2} - 2t^3 \times e^{-t^2}$

Set derivative equal to zero,

$e^{-t^2} \times 2t \times (1-t^2) = 0$

implies $t = 1$ for a positive solution
and from above analysis, this is clearly a max.

11. (i) $N_t \rightarrow$ No. of claims upto time 't'
Since poisson, $t = 0$.

$$P(T_0 \leq y \mid N_s = 1) = \frac{P(T_0 \leq y, N_s = 1)}{P(N_s = 1)}$$

$$= \frac{P(N_y = 1, N_s - N_y = 0)}{P(N_s = 1)}$$

But $N_s - N_y$ is independent of N_y
and has the same distribibtution as N_{s-y}.

Thus $RHS = \frac{(\lambda y e^{-\lambda y}) e^{-\lambda(s-y)}}{\lambda s e^{-\lambda s}} = \frac{y}{s}$

which is the CDF of uniform dist^n w/ par. $[0, s]$

(ii) $\lambda^n e^{-\lambda(t_1 + t_2 + \cdots t_n)} 1_{\{t_1, t_2, \ldots t_n > 0\}}$