Assignment -2

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49 A

1) Let claim sizes on a home insurance policy be represented by random variable 'X' and the car insurance policy be represented with 'Y'.

Given,

$$X \sim N(800, (100)^2) \quad \text{and,}$$

$$Y \sim N(1200, (300)^2)$$

Total number of home claims is $800 + X_1 + X_2 + X_3 + X_4$.

Total number of car claims is $Y_1 + Y_2 + Y_3$.

Also given that,

$$P(Y_1 + Y_2 + Y_3 > 800 + X_1 + X_2 + X_3 + X_4)$$

$$= P((Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) > 800)$$

It is known that if $Z = X \pm Y$ and $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$ then,

$$Z \sim N\left(\mu_1 \pm \mu_2 \, , \, \sigma_1^2 + \sigma_2^2\right)$$

Similarly,

$$P\left((Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) \, \cancel{800}\right)$$

$$\sim N\left((1200 + 1200 + 1200) - (800 + 800 + 800 + 800) \, , \, \left((300)^2 + (300)^2 + (300)^2\right) + \left((100)^2 + (100)^2 + (100)^2 + (100)^2\right)\right)$$

$$\sim N(400 \, , \, 310\,000)$$

Then, by converting to standard normal we have,

$$P\left((Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) > 800\right)$$

$$= P\left(Z > \frac{800 - 400}{(310000)^{\frac{1}{2}}}\right)$$

$$= P\left(Z > \frac{400}{556.776}\right) =$$

$$= 1 - P(Z < 0.718)$$

By referring to tables Pg:- 160, we have,

$P(Z < 0.71) = 0.76115$ and,

$P(Z < 0.72) = 0.76424$

By linear interpolation we have,

$P(Z < 0.718) = 0.763622$

Then, $P((Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) > 800) = 1 - 0.763622$

$= 0.236378$

2) Given, $X \sim \text{Gamma}(6,2)$

Y [illegible] $\sim$

~~N ~ Negative Binomial~~

Also given, $P(N=n) = {}^{n+2}C_n \, 0.9^3 \, 0.1^n$

$$= \frac{(n+2)!}{(n+2-n)!\, n!} 0.9^3 \, 0.1^n$$

$$= \frac{(n+2)(n+1)\cancel{n!}}{2!\,\cancel{n!}} (0.9)^3 (0.1)^n$$

$$= \frac{((n+3)-1)!}{n!\,(3-1)!} (0.9)^3 (0.1)^n$$

$$= \frac{\Gamma(n+3)}{\Gamma(n+1)\,\Gamma 3} (0.9)^3 (0.1)^n$$

Here, $k=3$ and $p=0.9$. i.e.,

$N \sim \text{Negative Binomial}(3, 0.9)$

(i)

Also given,

$$Y = X_1 + X_2 + X_3 + \dots + X_N .$$

Then, $\cancel{MGF_Y(t) = MGF}$ $\quad M_Y(t) = M_N[\log M_X(t)]$

Since, $X \sim$ Gamma$(6,2)$, and,

$$M_X(t) = \left(1 - \frac{t}{2}\right)^{-6} \Rightarrow \log M_X(t) = \log\left(1-\frac{t}{2}\right)^{-6} \quad \text{--- (1)}$$

Since, $N \sim$ Negative Binomial $(3, 0.9)$,

$$M_N(t) = \left(\frac{p}{1 - qe^t}\right)^k = \left(\frac{0.9}{1-(0.1)e^t}\right)^3$$

From (1), we have,

$$\cancel{M_X(t) = E(e^{tx}) =}$$

$$\Rightarrow M_N[\log M_X(t)] = \left(\frac{0.9}{1-(0.1)\exp\left(\log\left(1-\frac{t}{2}\right)^{-6}\right)}\right)^3$$

$$= \left(\frac{0.9}{1-(0.1)\left(1-\frac{t}{2}\right)^{-6}}\right)^3$$

(ii) $Var(Y) = E(N)\,Var(X) + var(N)\,[E(X)]^2$

$$= \left(\frac{3(0.1)}{0.9}\right)\left(\frac{6}{4}\right) + \left(\frac{3(0.1)}{0.81}\right)\left(\frac{6}{2}\right)^2$$

$$= 0.5 + \frac{10}{3}$$

$$= \frac{23}{6}$$

Standard Deviation $(Y) = 1.95789$

Date 07

3) (i) Given,

$$f_X(x) = \lambda e^{-\lambda(x-\alpha)} \quad ; \quad x \geq \alpha \text{ and } \lambda, \alpha > 0$$

Then, $M_X(t) = E\left[e^{tX}\right]$

$$= \int_{\alpha}^{\infty} \left(e^{tx}\right)\left(\lambda e^{-\lambda(x-\alpha)}\right) dx$$

$(\because x \geq \alpha)$

$$= \int_{\alpha}^{\infty} \lambda e^{[tx - \lambda(x-\alpha)]} dx$$

$$= \int_{\alpha}^{\infty} \lambda e^{tx - \lambda x + \lambda\alpha} dx$$

$$= \int_{\alpha}^{\infty} \lambda e^{\lambda\alpha} \cdot e^{-(\lambda-t)x} dx$$

$$= \lambda \cdot e^{\lambda\alpha} \left[-e^{-(\lambda-t)x} \cdot (\lambda-t)^{-1} \right]_{\alpha}^{\infty}$$

$$= \lambda \cdot e^{\lambda\alpha} \left[0 + e^{-\lambda\alpha + t\alpha} \cdot (\lambda-t)^{-1} \right]$$

$$= \lambda \cdot e^{\lambda\alpha} \cdot e^{-\lambda\alpha} \cdot e^{t\alpha} \cdot (\lambda - t)^{-1}$$

$$= \frac{\lambda}{\lambda - t} e^{t\alpha}$$

$$= \left(\frac{\lambda - t}{\lambda}\right)^{-1} e^{t\alpha}$$

$$M_x(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} e^{t\alpha}$$

(ii) $$M_x'(t) = e^{t\alpha}\left[\left(-1 \times \frac{-1}{\lambda}\right)\left(1 - \frac{t}{\lambda}\right)^{-2}\right]$$

$$+ \left(1 - \frac{t}{\lambda}\right)^{-1} e^{t\alpha} \cdot \alpha$$

$$M_x'(t) = e^{t\alpha} \times \frac{1}{\lambda} \cdot \left(1 - \frac{t}{\lambda}\right)^{-2} + \alpha\left(1 - \frac{t}{\lambda}\right)^{-1} e^{t\alpha}$$

$$\Rightarrow M_x'(0) = E(X) = e^{0 \times \alpha} \times \frac{1}{\lambda}\left(1 - \frac{0}{\lambda}\right)^{-2} + \alpha\left(1 - \frac{0}{\lambda}\right)^{-1} e^{0\alpha}$$

$$= \frac{1}{\lambda} + \alpha$$

$$M_X''(t) = e^{t\alpha}\left(2\times\frac{1}{\lambda}\times\frac{1}{\lambda}\left(1-\frac{t}{\lambda}\right)^{-3}\right) + \alpha\times\alpha\left(1-\frac{t}{\lambda}\right)^{-1}e^{t\alpha}$$

$$+ \; 2\alpha\times\frac{1}{\lambda}\left(1-\frac{t}{\lambda}\right)^{-2}e^{t\alpha}$$

$$M_X''(0) = E(X^2) = e^{0}\left(\frac{2}{\lambda^2}(1)^{-3}\right) + \alpha^2(1)^{-1}e^{0}$$

$$+ \; \frac{2\alpha}{\lambda}(1)^{-2}e^{0}$$

$$= \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2$$

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2 - \left(\frac{1}{\lambda}+\alpha\right)^2 = \frac{1}{\lambda^2}$$

4) Given,

$$G_V(t) = \left(\frac{p}{1-qt}\right)^{k}$$

It is a PGF of Negative Binomial type-2 distribution.

Therefore,

$$\cancel{P(V=v) = \begin{bmatrix} v-1 \\ k-1 \end{bmatrix} p^{k}}$$

$$P(V=v) = \frac{\Gamma(k+v)}{\Gamma(v+1)\,\Gamma(k)}\, p^{k} q^{v}$$

$$\Rightarrow P(V=4) = \frac{\Gamma(k+4)}{\Gamma 5\ \Gamma k}\, p^{k} q^{4}$$

$$= \frac{(k+3)!\,\cancel{p}}{4!\,(k-1)!}\, p^{k} q^{4}$$

(4) Given,

$$f_{X,Y}(x,y) = ax(x+y) \quad ; \quad 0<x<5, \; 10<y<20.$$

Then,

$$f_X(x) = \int_{10}^{20} ax(x+y)\,dy$$

$$= \int_{10}^{20} (ax^2 + axy)\,dy$$

$$= ax^2 [y]_{10}^{20} + \frac{ax}{2}[y^2]_{10}^{20}$$

$$= 10ax^2 + 150ax$$

$$f_Y(y) = \int_0^5 ax(x+y)\,dx$$

$$= \int_0^5 (ax^2 + axy)\,dx$$

$$= \frac{a}{3}[x^3]_0^5 + \frac{ay}{2}[x^2]_0^5$$

$$= \frac{125a}{3} + \frac{25ay}{2}$$

6) Given,

$X \sim Poi(\lambda)$ and $Y \sim Poi(\mu)$.

Then,

$M_X(t) = E[e^{tX}]$ and $M_Y(t) = E[e^{tY}]$

$= e^{\lambda(e^t - 1)}$ $= e^{\mu(e^t - 1)}$

(i) Let $Z = X + Y$. Then,

$$M_Z(t) = E[e^{tZ}] = E[e^{t(X+Y)}]$$

$$= E[e^{tX}] \cdot E[e^{tY}]$$

$$= M_X(t) \cdot M_Y(t)$$

$$= e^{\lambda(e^t - 1) + \mu(e^t - 1)}$$

$$M_Z(t) = e^{(\lambda+\mu)(e^t - 1)}$$

$\therefore Z \sim Poi(\lambda + \mu)$

$\Rightarrow X + Y \sim Poi(\lambda + \mu)$

(ii) Let $Z = X - Y$. Then,

$$M_Z(t) = E\left[e^{tZ}\right] = E\left[e^{t(X-Y)}\right]$$

$$= E\left[e^{tX}\right] \cdot E\left[e^{-tY}\right]$$

$$= \frac{E\left[e^{tX}\right]}{E\left[e^{tY}\right]}$$

$$= \frac{M_X(t)}{M_Y(t)}$$

$$= \frac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}}$$

$$\Rightarrow M_Z(t) = e^{(\lambda-\mu)(e^t-1)}$$

$\therefore Z \sim Poi(\lambda - \mu)$

$\Rightarrow X - Y \sim Poi(\lambda - \mu)$

7) (i) Here, We know that, $P(Y|X=x) = \dfrac{P(X=x, Y=y)}{P(X=x)}$

$X=x$	$X=x^2$	$P(X\|Y=2)$	$xP(X\|Y=2)$	$x^2P(X\|Y=2)$
1	1	$\frac{0}{0.6} = 0$	0	0
2	4	$\frac{0.3}{0.6} = 1/2$	1	2
3	9	$\frac{0.1}{0.6} = 1/6$	$1/2$	$9/6$
4	16	$\frac{0.2}{0.6} = 1/3$	$4/3$	$16/3$

$$E(X|Y=2) = \sum_x x P(X|Y=2)$$

$$= 0 + 1 + 1/2 + 4/3 = \frac{17}{6}$$

$$E(X^2|Y=2) = \sum_x x^2 P(X|Y=2)$$

$$= 0 + 2 + \frac{9}{6} + \frac{16}{3} = \frac{53}{6}$$

$$Var(X|Y=2) = E(X^2|Y=2) - [E(X|Y=2)]^2$$

$$= \frac{53}{6} - \left(\frac{17}{6}\right)^2 = \frac{29}{36}$$

(ii) Given,

$$f(u,v) = \frac{48}{67}(2uv - u^2) \quad ; \quad 0<u<1, \ \frac{u}{2}<v<2$$

Then, $f_V(v) = \int_{u} f(u,v)\,du$

$$= \int_0^1 \frac{48}{67}(2uv - u^2)\,du$$

$$= \frac{48}{67}\int_0^1 \cancel{(2uv)du} - u^2.$$

$$= \int_0^1$$

$$= \frac{48}{67}\left[\frac{2v}{2}\left[u^2\right]_0^1 - \frac{1}{3}\left[u^3\right]_0^1\right]$$

$$= \frac{48}{67}\times\left[\frac{2v}{2} - \frac{1}{3}\right]$$

$$E[U \mid V=v] = \cancel{\int_u u\, f(u|v)\, du} \quad \int_u u\, f(U \mid V=v)\, du$$

$$= \int_0^1 u \, \frac{f(u,v)}{f_V(v)}\, du$$

$$= \int_0^1 u \, \frac{\frac{48}{67}(2uv-u^2)}{\frac{48}{67}\left(v-\frac{1}{3}\right)}\, du$$

$$= \frac{1}{v-\frac{1}{3}} \int_0^1 2u^2v - u^3\, du$$

$$= \frac{1}{v-1/3}\left[\frac{2v}{3}\left[u^3\right]_0^1 - \frac{1}{4}\left[u^4\right]_0^1\right]$$

$$= \frac{1}{v-\frac{1}{3}}\left[\frac{2v}{3} - \frac{1}{4}\right]$$

8) Given,

$X \sim Gamma(3, 2)$ and,

$E(Y|X=x) = 3x+1$; $Var(Y|X=x) = 2x^2+5$

$E[X] = \frac{\alpha}{\lambda} = \frac{3}{2}$ $Var[X] = \frac{\alpha}{\lambda^2} = \frac{3}{4}$

We know that, $E[E[Y|X]] = E[Y]$ i.e

$$E[Y] = E[E[Y|X=x]]$$

$$= E[3x+1]$$

$$= 3E[X]+1$$

$$= \left(3 \times \frac{3}{2}\right)+1$$ ($\because E(X)$ when $X \sim Gamma(\alpha, \lambda)$ is $\frac{\alpha}{\lambda}$)

$$= 11/2$$

We also know that, $Var(Y) = E[Var[Y|X=x)] + Var[E(Y|X=x)]$

Then,

$$Var(Y) = E[2x^2+5] + Var[3x+1]$$

$$= 2E[X]+5 + 3^2 Var[X]$$

$$= \left(2 \times \frac{3}{2}\right)+5 + \left(9 \times \frac{3}{4}\right)$$

$$= \frac{59}{4}$$

$\boxed{S.D(Y) = \sqrt{\frac{59}{4}} = 3.8405}$

9) Given, $E[X] = 3$, $Var[X] = 4$

$E[Y] = 4$, $Var[Y] = 1$

$Corr(X,Y) = -0.3 = \dfrac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} \Rightarrow Cov(X,Y) = -0.6$

Let $Z = X+Y$.

(i) Then, $Y = Z - X$

And, $Cov(X,Z) = Cov(X, X+Y)$

$= Cov(X,X) + Cov(X,Y)$

$= E[X \cdot X] - E(X)E(X) + Cov(X,Y)$

$= E[X^2] - [E(X)]^2 + (-0.6)$

$= Var(X) - 0.6$

$= 4 - 0.6 = 3.4$

(ii) ~~Var(Z)~~ To find Var[Z].

$$Var[Z] = \cancel{Var}\ cov[Z, Z] = Cov[(X+Y), (X+Y)]$$

$$= Cov(X, X) + Cov(X, Y) + Cov(X, Y) + Cov(Y, Y)$$

$$= Var(X) + Var(Y) + \cancel{6}\ 2Cov(X, Y)$$

$$= 4 + 1 - 1.2$$

$$\Rightarrow Var[Z] = 3.8$$

10) Given,

$$f(x,y) = K x^{-\alpha} e^{-\frac{y}{\beta}} \quad ; 1<x<\infty, 1<y<\infty$$

We know that, $\int_x \int_y f(x,y)\, dx\, dy = 1$. Similarly,

$$\cancel{\int_1^\infty \int_1^\infty K x^{-\alpha} e^{-\frac{y}{\beta}} dx\, dy = 1}$$

$$\Rightarrow \int_1^\infty \int_1^\infty K x^{-\alpha} \cdot e^{-\frac{y}{\beta}} dx\, dy = 1$$

$$\Rightarrow K \int_1^\infty e^{-\frac{y}{\beta}} \int_1^\infty x^{-\alpha} dx\, dy = 1$$

$$\Rightarrow K \int_1^\infty e^{-\frac{y}{\beta}} \left[\frac{x^{-(\alpha-1)}}{-\alpha+1}\right]_1^\infty dy = 1$$

$$\Rightarrow \frac{-K}{-\alpha+1} \int_1^\infty e^{-\frac{y}{\beta}} dy = 1$$

$$\Rightarrow \frac{+K\beta}{-\alpha+1} \left[-e^{-\frac{y}{\beta}}\right]_1^\infty = 1$$

$$\Rightarrow K = \frac{\alpha-1}{\beta\,(e^{-1/\beta})}$$