23/11/21

CALCULUS ASSIGNMENT.

1. Evaluate the following limit.

$$\lim_{h\to 0} \frac{2(-3+h)^2-18}{h} = \lim_{h\to 0} \frac{2(9-6h+h^2)-18}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{18}-12h+2h^2-\cancel{18}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(-12+2h)}{\cancel{h}}$$

$$= -12+2(0)$$

$$= -12$$

2. Determine if the following function is continuous or discontinuous at (a) $x=4$, (b) $x=6$?

$$g(x) = \begin{cases} 2x & x<6 \\ x-1 & x\geq 6. \end{cases}$$

→ a) $x=4$

Put $x=4$ in $g(x)$ $\therefore g(4) = 2(4) = 8.$

$\therefore g(4)$ is defined.

$$\lim_{x\to 4^-} g(x) = \lim_{x\to 4} 2x = 2(4) = 8$$

$$\lim_{x\to 4^+} g(x) = \lim_{x\to 4^+} 2x = 2(4) = 8.$$

$$\therefore \lim_{x\to 4^-} = \lim_{x\to 4^+}$$

Hence $\lim_{x\to 4} g(x)$ exists

$$\therefore \lim_{x\to 4} g(x) = 8$$

$$g(4) = \lim_{x\to 4} g(x) = 8.$$

$\boxed{\therefore \text{The given function } g(x) \text{ is continuous at } x=4}$

b) $x = 6$

Put $x = 6$ in $g(x)$

$\therefore g(6) = 6 - 1 = 5$

$\therefore g(6) = 5$

$\therefore g(6)$ is defined.

$$\lim_{x \to 6^-} g(x) = \lim_{x \to 6^-} 2(x) = 12$$

$$\lim_{x \to 6^+} g(x) = \lim_{x \to 6^+} x - 1 = 6 - 1 = 5$$

$$\therefore \lim_{x \to 6^-} \neq \lim_{x \to 6^+} g(x).$$

$\therefore \lim_{x \to 6} g(x)$ does not exist

$\boxed{\text{$\therefore$ The given function $g(x)$ is discontinuous at $x = 6$}}$

3) Solve the following limit:

$$\lim_{x \to 9} \frac{x-9}{3-\sqrt{x}} = \lim_{x \to 9} \frac{x-9}{3-\sqrt{x}} \times \frac{3+\sqrt{x}}{3+\sqrt{x}}$$ ... (Rationalizing deno)

$$= \lim_{x \to 9} \frac{(x-9)(3+\sqrt{x})}{9-x}$$

$$= \lim_{x \to 9} \frac{-\cancel{(9-x)}(3+\sqrt{x})}{\cancel{9-x}}$$

$$= -3-\sqrt{9}$$

$$= -3-3$$

$$= \underline{\underline{-6}}$$

4. Determine if the following function is continuous or discontinuous at (a) $x=-1$ (b) $x=0$, (c) $x=3$?

$$f(x) = \frac{4x+5}{9-3x}$$

a) At $x=-1$, $f(-1) = \frac{4(-1)+5}{9-3(-1)} = \frac{1}{12}$

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \left(\frac{4x+5}{9-3x}\right) = \frac{1}{12}$$

$\boxed{\therefore f(x) \text{ is continous at } x=-1.}$

b) At $x=0$, $f(0) = \frac{4(0)+5}{9-3(0)} = \frac{5}{9}$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \left(\frac{4x+5}{9-3x}\right) = \frac{5}{9}$$

$\because f(0) = \lim_{a \to 0}$

$\boxed{\therefore \text{function is continuous at } x=0}$

c) At $x=3$, $f(3) = \frac{4(3)+5}{9-3(3)} = \frac{17}{0} = \infty$

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{4(3)+5}{9-3(3)} = \frac{17}{0} = \infty$$

$\because \lim_{x \to 3} = f(3)$

$\boxed{\therefore \text{function is continous at } x=3}$

5) Evaluate each of the following limits,

$$\lim_{x\to\infty} \frac{6e^{4x} - e^{-2x}}{8e^{4x} - e^{2x} + 3e^{-x}}$$

$$= \lim_{x\to\infty} \frac{e^{4x}\left[6 - e^{-6x}\right]}{e^{4x}\left[8 - e^{-2x} + 3e^{-5x}\right]} \qquad \left[\because \frac{e^{a}}{e^{b}} = e^{a-b}\right]$$

$$= \lim_{x\to\infty} \left[\frac{6 - 1/e^{6x}}{8 - 1/e^{2x} + 3/e^{5x}}\right]$$

$$= \cancel{\lim_{x\to\infty}} \frac{6 - 1/\infty}{8 - 1/\infty + 3/\infty}$$

$$= \frac{6}{8} = \frac{3}{4}$$

6) $g(y) = \begin{cases} y^2+5 & \text{if } y < -2 \\ 1-3y & \text{if } y \geq -2 \end{cases}$

a) $\lim_{x\to 6} g(y) = \lim_{y\to 6} 1-3y$

$= 1-3(6)$

$= 1-18$

$= -17.$

$$\boxed{\therefore \lim_{x\to 6} g(y) = -17}$$

b) $\lim_{x\to -2} g(y) = \lim_{x\to -2} 1-3y$

$= 1-3(-2)$

$= 1+6$

$= 7$

$$\boxed{\therefore \lim_{x\to -2} g(y) = 7}$$

7. Use the Product Rule to find the derivative of

$$f(t) = (4t^2 - t)(t^3 - 8t^2 + 12)$$

$$f'(t) = (4t^2 - t)\frac{d}{dt}(t^3 - 8t^2 + 12) + (t^3 - 8t^2 + 12)$$

$$\frac{d}{dt}(4t^2 - t)$$

$$= (4t^2 - t)(3t^2 - 16t) + (t^3 - 8t^2 + 12)(8t - 1)$$

$$= 12t^4 - 64t^3 - 3t^3 + 16t^2 + 8t^4 - 64t^3 + 96t - t^3 + 8t^2 - 12.$$

$$= 20t^4 - 132t^3 + 24t^2 + 96t - 12.$$

$$\boxed{\therefore f'(t) = 20t^4 - 132t^3 + 24t^2 + 96t - 12}$$

8. Use the Quotient Rule to find the derivative

$$R(\omega) = \frac{3\omega + \omega^4}{2\omega^2 + 1}$$

Differentiate w.r.t. ω.

$$R'(\omega) = \frac{(2\omega^2 + 1)\frac{d}{d\omega}(3\omega + \omega^4) + (3\omega + \omega^4)\frac{d}{d\omega}(2\omega^2 + 1)}{(2\omega^2 + 1)^2}$$

$$= \frac{(2w^2+1)(3+4w^3)+(3w+w^4)(4w)}{(2w^2+1)^2}$$

$$= \frac{6w^2+8w^5+3+4w^3+12w^2+4w^5}{(2w^2+1)^2}$$

$$= \frac{12w^5+4w^3+18w^2+3}{(2w^2+1)^2}$$

$$\boxed{\therefore R'(w) = \frac{12w^5+4w^3+18w^2+3}{(2w^2+1)^2}}$$

9. Differentiate;

$f(x) = 2e^x - 8^x$

Differentiating w.r.t. x.

$\boxed{\therefore f'(x) = 2e^x - 8^x \cdot \log 8}$

10. Differentiate;

$y = z^5 - e^z \ln(z)$

Differentiating w.r.t. ~~y~~ z

$\cancel{\therefore \frac{dy}{dx} = z^5 - e^z \ln(z)}$

$$\frac{dy}{dz} = 5z^4 - \left[e^z \frac{d}{dz}\ln(z) + \ln(z)\frac{d}{dz}(e^z)\right]$$

$$\frac{dy}{dz} = 5z^4 - \left[e^z \times \frac{1}{z} + \ln(z)\, e^z\right]$$

$$\boxed{\therefore \frac{dy}{dz} = 5z^4 - e^z\left(\frac{1}{z} + \ln(z)\right)}$$

11. Using Chain Rule Differentiate the following:

a) $f(x) = (6x^2 + 7x)^4$ $\quad \therefore y = (6x^2+7x)^4$

Putting $u = 6x^2 + 7x$ we get

~~f'(x) =~~

$$f(x) = (u)^4 = y$$

$$\therefore \frac{dy}{du} = 4u^3 \quad \text{and}$$

$$\frac{du}{dx} = 12x + 7.$$

$$\therefore \frac{dy}{du} = 4(6x^2+7x)^3 \quad \text{and} \quad \frac{du}{dx} = 12x+7$$

By Chain Rule, we get

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 4(6x^2+7x)^3(12x+7)$$

$$\boxed{\therefore f'(x) = 4(6x^2+7x)^3(12x+7)}$$

~~12. Determine the second derivate of~~

b) $f(t) = 5 + e^{4t+t^7}$

$$f'(t) = 5 + e^{4t+t^7} \frac{d}{dt}(4t+t^7)$$

$$\boxed{\therefore f'(t) = 5 + e^{4t+t^7}(4+7t^6)}$$

12. Determine the second derivate of;

a) $z = \ln(7-x^3)$

$z' = \frac{1}{7-x^3} \frac{d}{dx}(7-x^3)$

$z' = \frac{1}{7-x^3}(-3x^2)$

$z' = \frac{-3x^2}{7-x^3}$

$z'' = \frac{(7-x^3)\frac{d}{dx}(-3x^2) + (-3x^2)\frac{d}{dx}(7-x^3)}{(7-x^3)^2}$

$z'' = \frac{(7-x^3)(-6x) + (-3x^2)(-3x^2)}{(7-x^3)^2}$

$z'' = \frac{-42x + 6x^4 + 9x^4}{(7-x^3)^2}$

$z'' = \frac{15x^4 - 42x}{(7-x^3)^2}$

$\boxed{\therefore z'' = \frac{15x^4 - 42x}{(7-x^3)^2}}$

b) $Q(v) = \frac{2}{(6+2v-v^2)^4}$

$Q'(v) = \frac{\cancel{2}}{{}_{2}4(6+2v-v^2)^3(2-2v)}$

$Q''(v) = \frac{1}{(6+2v-v^2)^3 \frac{d}{dv}(2-2v) + (2-2v)\frac{d}{dv}(6+2v-v^2)^3}$

$= \frac{1}{(6+2v-v^2)^3(-2) + (2-2v)\,3(6+2v-v^2)^2(2-2v)}$

$= \frac{1}{-2(6+2v-v^2)^3 + 3(2-2v)^2(6+2v-v^2)^2}$

c) $f(t) = \ln(1+t^2)$

$$f'(t) = \frac{1}{1+t^2} \cdot \frac{d}{dt}(1+t^2)$$

$$f'(t) = \frac{2t}{1+t^2}$$

$$f''(t) = \frac{(1+t^2)\frac{d}{dt}(2t) + 2t\frac{d}{dt}(1+t^2)}{(1+t^2)^2}$$

$$f''(t) = \frac{(1+t^2)(2) + (2t)(2t)}{(1+t^2)^2}$$

$$f''(t) = \frac{2+2t^2+4t^2}{(1+t^2)^2}$$

$$\boxed{f''(t) = \frac{2+6t^2}{(1+t^2)^2}}$$

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13. Find the maximum and minimum value of the function:

$f(x) = x^3 - 3x^2 - 9x + 12.$

$\therefore$ $f'(x) = 3x^2 - 6x - 9$ and

$f''(x) = 6x - 6$

Now, $f'(x) = 0.$

$3(x^2 - 2x - 3) = 0$

$3[(x^2 - 3x - x - 3)] = 0$

$3[x(x-3) - 1(x-3)] = 0$

$3(x-3)(x-1) = 0$

$\therefore$ $x = 3$ or $x = 1$

For $x = 3$,

$f''(1) = 18 - 6 = 12$.

$\therefore$ $f(x)$ attains ~~maxi~~ minimum at $x = 3 > 0$

$\therefore$ Minimum Value $= f(3) = (3)^3 - 3(3)^2 - 9(3) + 12$

$= 27 - 27 - 27 + 12$

$= -15$

$\therefore$ The function $f(x)$ has minimum value -15 at $x = 3$.

14. Find the maximum and minimum value of f^n:

$f(x) = 4x^3 - 18x^2 + 24x - 7$

$\therefore f'(x) = 12x^2 - 36x + 24$

$f''(x) = 24x - 36$

Now, $f'(x) = 0$

$12(x^2 - 3x + 2) = 0$

$12(x^2 - 2x - x + 2) = 0$

$12[x(x-2) - (x-2)] = 0$

$12(x-2)(x-1) = 0$

$\therefore x = 2$ or $x = 1$.

For $x = 1$,

$f''(1) = 24(1) - 36 = -12 < 0$

$\therefore f(x)$ attains maximum at $x = 1$

$\therefore$ Maximum value $= f(1)$

$= 4(1)^3 - 18(1)^2 + 24(1) - 7$

$= 4 - 18 + 24 - 7$

$= 3$.

$\therefore$ The function $f(x)$ has maximum value 3 at $x = 1$.

For $x = 2$,

$f''(2) = 24(2) - 36$

$= 48 - 36$

$= 12 > 0$

$\therefore f(x)$ attains minimum at $x = 2$

Minimum value $= f(2)$

$= 4(2)^3 - 18(2)^2 + 24(2) - 7$

$= 32 - 72 + 48 - 7$

$= 1$

$\therefore$ The function $f(x)$ has minimum value 1 at $x = 2$.

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15. Sketch the curve for the following function;

$f(x) = x^2(x+3)$

$f'(x) = x^3 + 3x^2$

Differentiating $f(x)$ w.r.t. x

$f'(x) = 3x^2 + 6x$.

Differentiating $f'(x)$ w.r.t. x

$f''(x) = 6x + 6$

When $f'(x) = 0$

$3x^2 + 6x = 0$

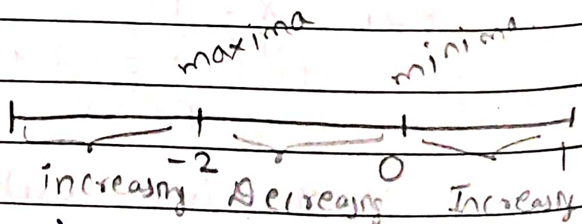
$\therefore 3(x^2 + 2x) = 0$

$\therefore x^2 + 2x = 0$

$\therefore x(x+2) = 0$

$\therefore x = 0$ or $(x+2) = 0$

$\therefore x = 0$ or $x = -2$.



Put $x = 0$ in $f''(x)$

$\therefore f''(0) = 6 > 0$

Put $x = -2$ in $f''(x)$

$\therefore f''(-2) = -6 < 0$

$\therefore$ The given function is maximum at $x = -2$ and minimum at $x = 0$

Put $x = -1$ in $f'(x)$

$\therefore f'(-1) = 3(-1)^2 + 6(-1)$

$= 3 - 6$

$= -3 < 0$

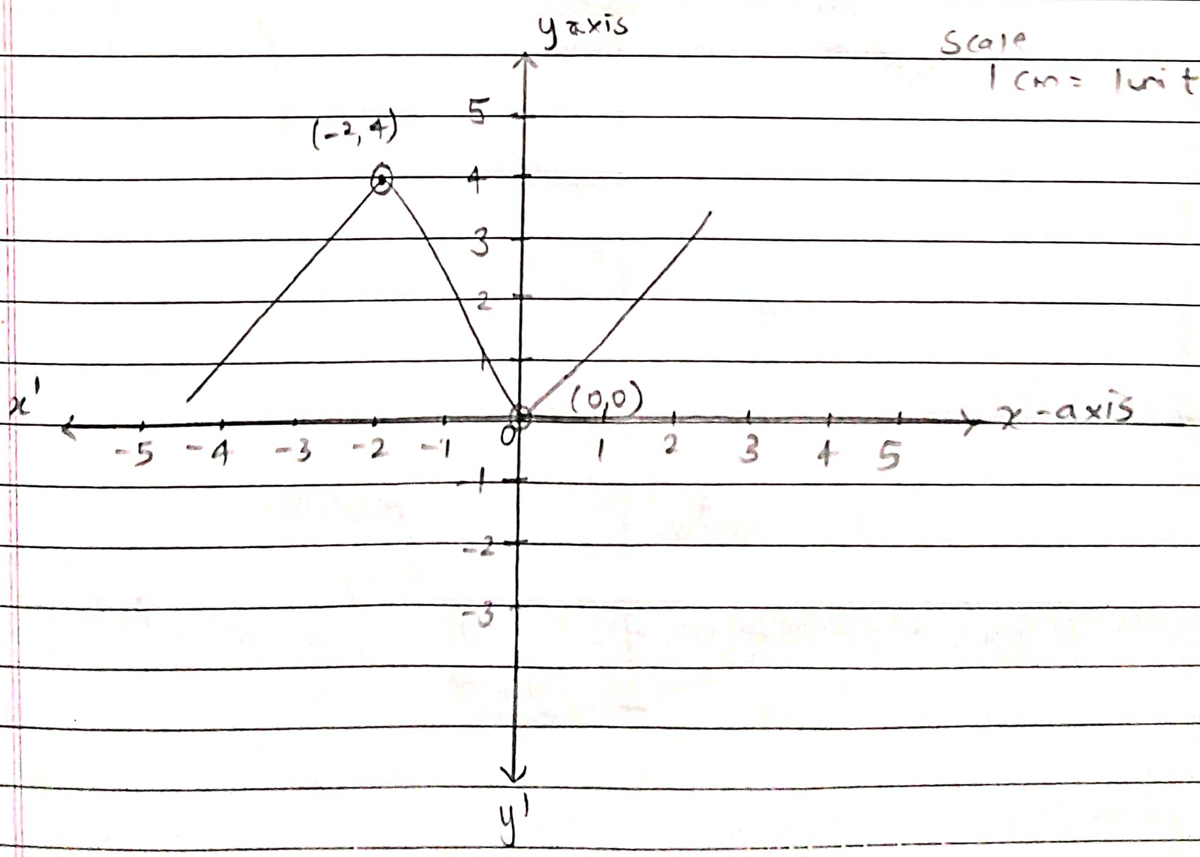
$\therefore$ The given function is decreasing between -2 and 0.

Put $x = 0$ in $f(x)$

Minimum value $= f(0) = 0(0+3) = 0$ $\therefore x = 0, y = 0$

Put $x = 2$ in $f(x)$

Maximum value $= f(-2) = (-2)^2(-2+3) = 4$. $\therefore x = -2, y = 4$



16) Sketch a detailed graph for the following function

$f(x) = 3x^2 - 6x$

i.e. $y = 3x^2 - 6x$

Put $x = 0$ in $f(x)$

$\therefore y = 3(0)^2 - 6(0)$

$\therefore y = 0$

Put $x = 1$ in $f(x)$

$\therefore y = 3(1)^2 - 6(1)$

$\therefore y = -3$

Put $x = -1$ in $f(x)$

$\therefore y = 3(-1)^2 - 6(-1)$

$\therefore y = 9$

Put $x = 2$ in $f(x)$

$\therefore y = 3(2)^2 - 6(2)$

$\therefore y = 0$

Put $x = -2$ in $f(x)$

$\therefore y = 3(-2)^2 - 6(-2)$

$\therefore y = 24$

Put $x = 3$ in $f(x)$

$\therefore y = 3(3)^2 - 6(3)$

$\therefore y = 9$

Put $x = -3$ in $f(x)$

$\therefore y = 3(-3)^2 - 6(-3)$

$\therefore y = 45$

x	y
0	0
1	-3
-1	9
2	0
-2	24
3	9
-3	45

Scale
1cm = 5 unit on y-axis
1cm = 1 unit on x-axis