FINANCIAL ENGENEERING - 1
ASSIGNMENT

1\. i) An arbitrage opportunity involves taking advantage of market inefficiencies to make a risk less trading profit
A riskless trading profit means (i) an immediate profit withe zero probability of future loss or (ii) zero cost with a zero probability of future loss and a non zero probability of future profit

(ii) The law of one price states that if there are two securities with the same profit or loss, then their prices should be equal otherwise there will be arbitrage

(iii) $S_0 = 125$, $K = 120$, $C = 30$, $r = 5\%$ p.a., $t = 3$ months

a) $q = 15\%$ p.a.
Using the Put Call Parity:

$$S_0 e^{-qt} + pt = c + Ke^{-rt}$$

$$pt = 30 + 120 e^{-0.05 \times 3/12} - 125 e^{-0.15 \times 3/12}$$

$$= 28.11$$

3.

(i) $f(t,x) = e^{-t}x^2$

$\frac{df}{dt} = -e^{-t}x^2 = -f$

$\frac{df}{dx} = e^{-t}.2x = 2e^{-t}x$

$\frac{df}{dx^2} = 2e^{-t}(1) = 2e^{-t}$

Let $Y_t = f(t, X_t)$

Applying Ito's Lemma:

$dY_t = \frac{df}{dt}dt + \frac{df}{dx}dX_t + \frac{1}{2}\frac{d^2f}{dx^2}\sigma^2 X_t^2 dt$

$= -f\,dt + 2e^{-t}X_t\,dX_t + e^{-t}\sigma^2 X_t^2 dt$

$= -Y_t\,dt + 2e^{-t}X_t^2\frac{dX_t}{X_t} + e^{-t}\sigma^2 X_t^2 dt$

$= -Y_t\,dt + 2Y_t[0.25dt + \sigma\,dW_t] + \sigma^2 Y_t\,dt$

$\therefore \frac{dY_t}{Y_t} = [2(0.25) - 1 + \sigma^2]dt + 2\sigma\,dWt$

$= [\sigma^2 - 0.5]dt + 2\sigma\,dWt$

$\therefore dY_t = [\sigma^2 - 0.5]Y_t\,dt + 2\sigma Y_t\,dWt$

(ii) The process is a martingale when drift is 0

$\therefore \sigma^2 - 0.5 = 0$

$\therefore \sigma^2 = 0.5$

4. Let d_1 be the dividend at time t_1 before the stock goes ex-dividend, d_2 be the dividend at time t_2 and so on until d_n as the dividend for time t_n

If the call option is exercise befor the ex-dividend date, then the payoff $= S(t_n) - K$

If the option is not exercised, the holder of the option receives $S(t_n) - d_n$

It is never optimal to exercise the option if:

$S(t_n) - d_n - Ke^{-r(T-t_n)} > S(t_n) - K$

$\therefore d_n < K(1-e^{-r(T-t_n)})$

$\therefore K(1-e^{-r(T-t_n)}) = 350(1-e^{-0.95(10/12 - 3/12)})$

$= 18.87$

$\therefore K(1-e^{-r(T-t_n)}) = 65(1-e^{-0.95(10/12 \times 3/12)})$

$= 10.91$

Thus, it is never optimal to exercise the american option on the two ex-dividend dates

5. The stock price follows a Geometric Brownian Motion

$$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}$$

$$\therefore \ln(S_t) \sim N\left[\ln(S_0) + (\mu - \sigma^2/2)t,\ \sqrt{\sigma^2 t}\right]$$

$$\therefore \ln(S_t) \sim N\left[\ln(S_0) + (\mu - \sigma^2/2)t,\ \sigma t^{1/2}\right]$$

$$S_0 = 254,\ \sigma = 0.35,\ \mu = 0.16,\ t = 0.5$$

$$\therefore \ln(S_t) \sim N\left[\ln(254) + (0.16 - 0.35^2/2)(0.5),\ 0.35(0.5)^{1/2}\right]$$

$$\therefore \ln(S_t) \sim N[5.59,\ 0.247]$$

$$\therefore \frac{\ln(S_t) - \mu(S_t)}{\sigma t^{1/2}} \sim \underbrace{N(0,1)}_{\longrightarrow \text{ standard normal distribution}}$$

$$\therefore P(S_t > 258) = 1 - \phi\left[\frac{\ln(258) - \mu(S_t)}{\sigma t^{1/2}}\right]$$

$$= 1 - \phi\left[\frac{5.55 - 5.59}{0.247}\right]$$

$$= 1 - \phi(-0.1364)$$

$$= 0.5542$$

But, the put option is exercised if $S_t < 258$ in 6 months

$$\therefore \text{Probability} = 1 - 0.5542$$

$$= 0.4457$$

6.(i) $\log(S_t / S_0) = \mu t + \sigma B_t$

$\therefore S_t = S_0 e^{\mu t + \sigma B_t}$

Applying Ito's Lemma :

$dB_t = 0 \times dt + 1 \times dB_t$

Let $G(t, B_t) = S_t = S_0 e^{\mu t + \sigma B_t}$

$\therefore \frac{dG}{dt} = \mu S_0 e^{\mu t + \sigma B_t} = \mu S_t$

$\therefore \frac{dG}{dB_t} = \sigma S_0 e^{\mu t + \sigma B_t} = \sigma S_t$

$\therefore \frac{d^2 G}{dB_t^2} = \sigma^2 S_t$

$\therefore dG = [0 \times dS_t + \frac{1}{2} \times 1^2 \times \sigma^2 S_t + \mu S_t] dt + 1 \times S_t dB_t$

$\therefore dS_t = (\mu + \frac{1}{2}\sigma^2) S_t \, dt + \sigma S_t dB_t$

$\therefore dS_t / S_t = \sigma dB_t + (\mu + \frac{1}{2}\sigma^2) dt$

$\therefore c_1 = \sigma$

$\therefore c_2 = \mu + \frac{1}{2}\sigma^2$

(ii) $E[S_t] = E[S_0 e^{\mu t + \sigma B_t}]$

$= S_0 e^{\mu t} E[e^{\sigma B_t}]$

since $B_t \sim N(0,1)$: $E[e^{tB_t}] = e^{1/2 t \cdot 2t}$

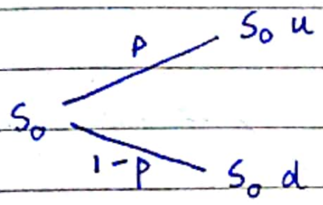
$\therefore E[S_t] = S_0 e^{\mu t} . e^{1/2 \sigma 2t}$

$= S_0 e^{\mu t + 1/2 \sigma 2t}$

$$Var[S_t] = E[S_t^2] - [E(S_t)]^2$$
$$= E[S_0^2 e^{2\mu t + 2\sigma B t}] - (S_0 e^{\mu t + 1/2 \sigma 2t})^2$$
$$= S_0^2 e^{2\mu t} E[e^{2\sigma B t}] - S_0^2 e^{2\mu t + \sigma 2t}$$
$$= S_0^2 e^{2\mu t + 2\sigma 2t} - S_0^2 e^{2\mu t + \sigma 2t}$$
$$= S_0^2 e^{2\mu t}[e^{2\sigma 2t} - e^{\sigma 2t}]$$

7.

(i) $p \rightarrow$ probability of an upmove



$$\therefore E[C_t] = S_0[pu + (1-p)d] \quad — ①$$

$$\therefore Var[C_t] = S_0^2[pu^2 + (1-p)d^2] - S_0^2[pu + (1-p)d]^2$$
$$= S_0^2[pu^2 + (1-p)d^2 - (pu + (1-p)d)^2]$$

$E[C_t^2] - E[C_t]^2$

Assuming $d = 1/u$

$$\therefore Var[C_t] = S_0^2[p(1-p)u^2 + p(1-p)d^2 - 2p(1-p)]$$
$$= S_0^2\, p(1-p)(u-d)^2 \quad — ②$$

From ① : $S_0 e^{rt} = S_0[pu + (1-p)d]$ — ③

From ② : $\sigma^2 S_0^2 t = S_0^2\, p(1-p)(u-d)^2$ — ④

$$\therefore p = \frac{e^{rt} - d}{u - d}$$

Substituting p in ④ :

$$\sigma^2 t = \frac{e^{rt} - d}{u-d}\left(1 - \frac{e^{rt} - d}{u-d}\right)(u-d)^2$$

$$\sigma^2 t = -(e^{rt} - d)(e^{rt} - u)$$

$$\therefore \sigma^2 t = (u+d)e^{rt} - (1 + e^{2rt})$$

$d = 1/u$, multiplying throughout by u

$$\therefore u^2 e^{rt} - u(1 + e^{2rt} + \sigma^2 t) + e^{rt} = 0$$

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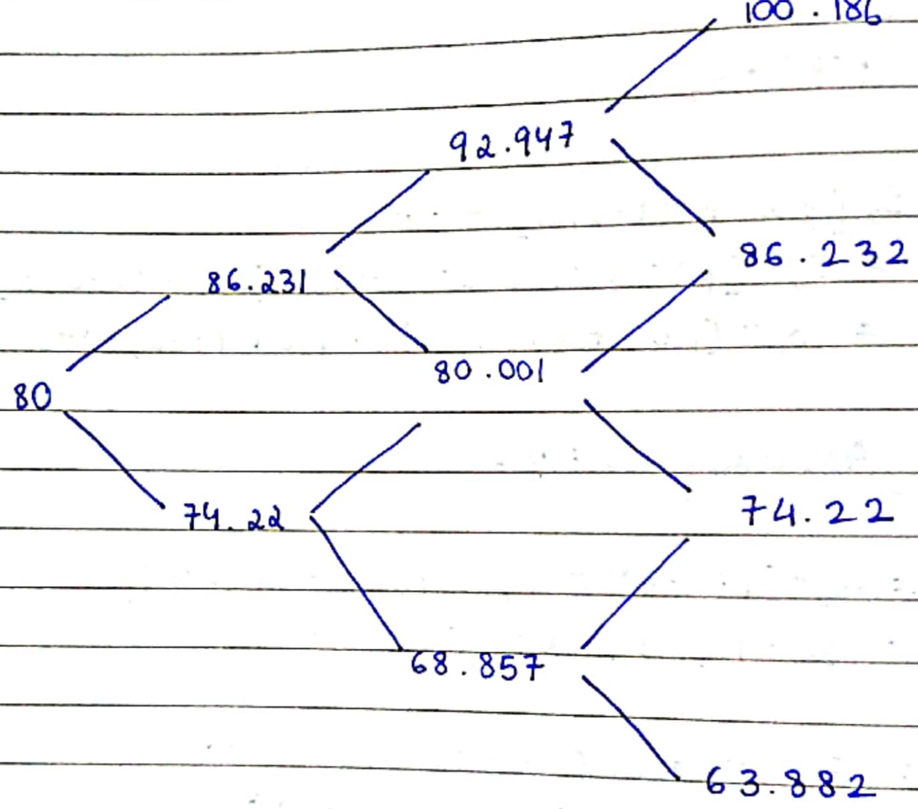
(ii) $\sigma = 0.15$, $t = \frac{3}{12} = 0.25$ years

$\therefore$ $u = e^{\sigma\sqrt{t}}$

$\therefore$ $u = 1.077884$

$\therefore$ $d = 1/u = 0.92774$

a)



8.

i) Let the current stock price $= S_0$

Let the current option price $= C_0$

Let the time to maturity $= T$

Value at upnode $= S_0 . u$

Value at downnode $= S_0 . d$

Let the payoff $= C_u$ if the $S_t = S_0 u$

$= C_d$ if the $S_t = S_0 d$

Consider a portfolio where we sell the option and by 'K' shares, where the value of K will make the portfolio risk free

For an upward movement: portfolio $= KS_0u - C_u$

For a downward movement : portfolio $= KS_0d - C_d$

For the two to be equal : $KS_0u - C_u = KS_0d - C_d$

$K = \dfrac{C_u - C_d}{S_0u - S_0d}$ makes the portfolio risk free at r rate of interest

Present Value $= (KS_0u - C_u)e^{-rT}$

Cost of portfolio $= KS_0 - C_0$

$\therefore (KS_0u - C_u)e^{-rT} = KS_0 - C_0$

$\therefore C_0 = KS_0 - [KS_0u - C_u]e^{-rT}$

$\therefore C_0 = KS_0 - KS_0ue^{-rT} + C_ue^{-rT}$

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(ii) The probability of an upward movement in stock increases the value of a call option.
A decrease in the probability of the stock price going down decreases the value of a put option

(iii) $E[S_t]$ at time $T = pS_0u + (1-p)S_0d$
where p : probability of an upmove

$$\therefore \; p = \frac{e^{rT} - d}{u-d}$$

$$\therefore \; E[S_t] = S_0 e^{rt}$$
$\hookrightarrow$ stock price grows at the risk free rate of interest

(iv) in a risk neutral would, investors do not require compensation for risk and hence the expected return is the risk free rate of return.

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9.

i) Forward Price $= F = S_0 e^{rt}$

Put Call Parity : $c + Ke^{-rt} = p + S_0$

$t = 3$ months $= 0.25$ years

$13.334 + 70e^{-0.25r} = 0.12 + S_0$ — ①

$8.869 + 75e^{-0.25r} = 0.568 + S_0$ — ②

$\therefore S = 82, \ r = 0.07 = 7\%$

$\therefore F = S_0 e^{rt} = 83.447629$

$= 83.4476$

ii) Using the Put Call Parity for each variable:

$6.899 + a \cdot e^{-(0.07 \times 0.25)} = 1.055 + 82$

$\therefore a = 77.5$

$b + 80e^{-(0.07 \times 0.25)} = 1.789 + 82$

$\therefore b = 5.177$

$2.594 + 85e^{-(0.07 \times 0.25)} = c + 82$

$\therefore c = 4.119$

10.

i) $q = \frac{e^r - d}{u - d}$

Since $r = 0\%$: $q = \frac{1-d}{u-d}$

Assuming : $d = 1/u$ and substituting :

$q = [1 - \frac{1}{u}] / [u - \frac{1}{u}]$

$q = \frac{u-1}{u} \times \frac{u}{u^2-1}$

$q = \frac{u-1}{(u-1)(u+1)}$

$\therefore$ $q = \frac{1}{u+1}$

For No Arbitrage : $d < e^r < u$

$d < e^0 < u$

$d < 1 < u$

$\therefore$ $u > 1$

Adding 1 on both sides :

$u+1 > 1+1$

$u+1 > 2$

$\therefore$ $\frac{1}{u+1} < \frac{1}{2}$

$\therefore$ $q < \frac{1}{2}$

Hence Proved

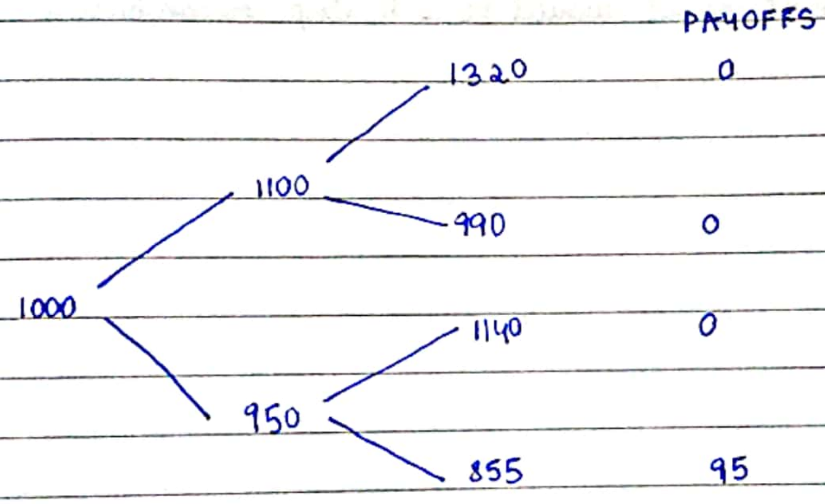
11. i) A recombining binomial model assumes that the underlying random variable or underlying asset is a discrete process. The upsteps (u) and downsteps (d) are the same across all time intervals. The risk neutral probability (q) also remains constant.

→ A N-step recombining model gives us N+1 outcomes as opposed to 2N possible outcomes in a non-recombining model, which greatly reduces the time required for computation.

→ However, the recombining model assumes that the drift and volatility parameters of the underlying asset are constant, which is not a valid assumption based on evidence.

ii)

a)



$$q_1 = \frac{e^{r} - d}{u-d} = \frac{e^{0.0175} - 0.95}{1.10 - 0.95} = 0.4510$$

$$q_2 = \frac{e^{r} - d}{u-d} = \frac{e^{0.025} - 0.90}{1.20 - 0.90} = 0.41772$$

Value at Node 1 (up) $= [0 \times q_2 + 0 \times (1-q_2)]\, e^{-r}$
$= 0$

Value at Node 1 (down) $= [0 \times q_2 + 95\,(1-q_2)]\, e^{-r}$
$= 53.95082828$

$\therefore$ Price of the option $= \{[0 \times q_1] + [53.9508 \times (1-q_1)]\} e^{-r}$
$= 53.9508\,(1-q_1) \times e^{-0.0175}$
$= 29.105$

b) The proposed model would have had to have values of u and d for each branch as well as a separate risk-free rate of interest.
This tree would have $2^N = 2^6 = 24$ nodes, making computation difficult and more time consuming
A more efficient model would be a 6 step recombining tree

12. $Z(t)$ is a standard Brownian Motion

a) $U(t) = 2Z(t) - 2$

$\therefore dU(t) = 2dZ(t) - 0$

$\therefore dU(t) = 0\,dt + 2dZ(t)$

$\therefore$ The process $U(t)$ has zero drift

b) $V(t) = [Z(t)]^2 - t$

$dV(t) = d[Z(t)]^2 - dt$

$dV(t) = 2dZ(t)\,Z(t) + 2/2\,[dZ(t)]^2$

$dV(t) = 2Zt\,dZ(t) + dt$

$\therefore dV(t) = 2Z(t)\,dZ(t)$

$\therefore$ The process $V(t)$ has zero drift

c) $W(t) = t^2 Z(t) - 2\int_0^t s\,Z(s)\,ds$

$dW(t) = d[t^2 Z(t)] - 2t\,Z(t)\,dt$

$dW(t) = t^2 dZ(t) + 2tZ(t)dt - 2tZ(t)dt$

$\therefore dW(t) = t^2 dZ(t)$

$\therefore$ The process $W(t)$ has zero drift

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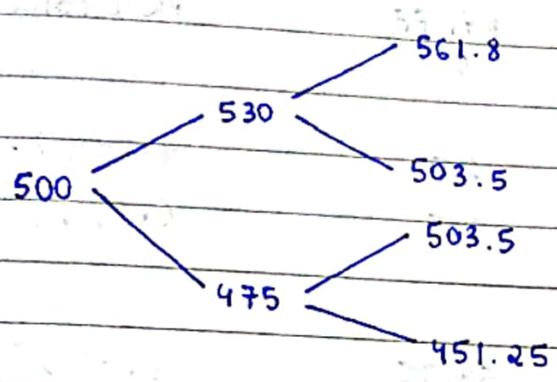
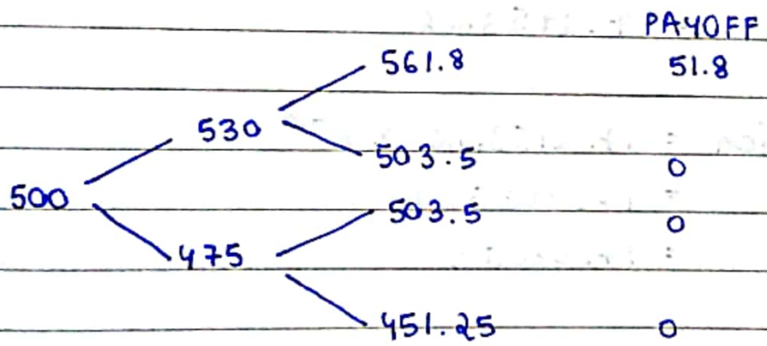
14. $S_0 = 500$, $u = 6\% \rightarrow 1.06$, $d = 5\% \rightarrow 0.95$, $r = 5\%$ p.a. $= \frac{0.05}{12} \times 3$ per 3 months



$$q = \frac{e^{r} - d}{u - d} = 0.568895014$$

(i) For a European call option : $K = 510$



Payoff $\times$ Probability $= 51.8 \times q^2 \times {}^2C_2$
$= 16.76463161$

$\therefore$ Price of the call option $= 16.764631 \times e^{-r}$
$= 16.556378$
$= 16.56$

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15 a (i) Let $f = f(S_t, t) = S^k$

$$\frac{df}{dS} = k.S^{k-1}$$

$$\frac{d^2f}{dS^2} = k(k-1)S^{k-2}$$

$$\frac{df}{dt} = 0$$

Using Ito's Lemma:

$$df = kS^{k-1}dS_t + \frac{1}{2}k(k-1)S^{k-2}(dS_t)^2$$

$$= f[k\mu + \frac{1}{2}k(k-1)\sigma^2]dt + f[k\sigma]dW_t$$

Hence $f = S^k$ follows a Geometric Brownian Motion

drift $= k\mu + \frac{1}{2}k(k-1)\sigma^2 = \mu'$

volatility $= k\sigma = \sigma'$

$\therefore f_t = f_0 e^{(\mu t - 1/2\sigma^2 t) + \sigma W t}$

$\therefore \frac{f_t}{f_0} \sim \log N\left((\mu' - 1/2\,\sigma'^2)t, \sigma'^2 t\right)$

$\therefore E[(f)] = f_0 e^{\mu' t}$

$\therefore V[f] = f_0^2 e^{2\mu' t}(e^{\sigma'^2 t} - 1)$

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(ii) Let $f = f(t, S_t) = e^{-rt} S_t$

$\frac{df}{dt} = -re^{-rt} S_t$

$\frac{df}{dS} = e^{-rt}$

$\frac{d^2f}{dS^2} = 0$

Using Ito's Lemma:

$df = -re^{-rt} S_t \, dt + e^{-rt} dS_t$

$= (\mu - r) f \, dt + \sigma f \, dW_t$

df is a martingale only when $\mu = r$

(iii) discounted S^k will be a martingale:

$k\mu + \frac{1}{2}k(k-1)\sigma^2 = r$

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