P & S Assignment - 2

1) $H \sim N(800, 100^2)$

$C \sim N(1200, 300^2)$

$$P((C_1 + C_2 + C_3) > (H_1 + H_2 + H_3 + H_4 + 800))$$

$$= P((C_1 + C_2 + C_3) - (H_1 + H_2 + H_3 + H_4) > 800)$$

$$\Rightarrow (C_1 + C_2 + C_3) - (H_1 + H_2 + H_3 + H_4) \sim N(1200 \times 3 - 4 \times 800, \; 3 \times 300^2 + \cancel{3 \times 30} \; 4 \times 100^2)$$

$$\Rightarrow (C_1 + C_2 + C_3) - (H_1 + H_2 + H_3 + H_4) \sim N(400, 310000)$$

$$P((C_1 + C_2 + C_3) - (H_1 + H_2 + H_3 + H_4) > 800) = P\left(Z > \frac{800 - 400}{\sqrt{310000}}\right)$$

$$= P(Z > 0.718)$$

$$= 1 - P(Z < 0.718)$$

$$= 0.236$$

$\sim\infty$.

2) $N \sim$ Negative Binomial

$$P(N=n) = {}^{n+2}C_n (0.9)^3 \times (0.1)^n$$

$X \sim$ Gamma $(6,2)$, $Y = X_1 + X_2 + \dots + X_N$

~~i) $M_Y(t) = E(e^{tY})$~~

$$M_N(t) = \left(\frac{pe^t}{1-qe^t}\right)^3 = \left(\frac{0.9e^t}{1-0.1e^t}\right)^3$$

$$M_X(t) = \left(1-\frac{t}{2}\right)^{-6}$$

1) $M_Y(t) = E(e^{tY}) = E(E(e^{tY}/N))$

$E(e^{tY}/N) = E(e^{t(X_1+X_2+\dots X_N)}/N)$

$= E(e^{t(X_1+X_2+\dots X_n)})$

$= E(e^{tX_1}).E(e^{tX_2})\dots E(e^{tX_n})$

$= M_{X_1}(t).M_{X_2}(t)\dots M_{X_n}(t)$

$= [M_X(t)]^n$

$M_Y(t) = E(E(e^{tY}/N))$

$= E((M_X(t))^n)$

$= E(e^{\ln[M_X(t)]^n}) = E(e^{n\ln[M_X(t)]})$

$M_Y(t) = M_N(\ln(M_X(t)))$

ii) $M_Y(t) = M_N\left[\ln\left(1-\frac{t}{2}\right)^{-6}\right] = M_N\left[-6\ln\left(1-\frac{t}{2}\right)\right]$

$= \left(\frac{0.9e^t}{1-0.1e^t}\right)^3\left(-6\ln\left(1-\frac{t}{2}\right)\right)$

3) i) $M_X(t) = E(e^{tX}) = \int_{\alpha}^{\infty} e^{tx} \lambda e^{-\lambda(x-\alpha)} \cdot dx$

$\left(f(x) = \lambda e^{-\lambda(x-\alpha)}\right)$

$$= \int_{\alpha}^{\infty} \lambda e^{\lambda\alpha} \cdot e^{-(\lambda-t)x} \cdot dx$$

$$= \left[-\frac{\lambda e^{\lambda\alpha}}{\lambda-t} e^{-(\lambda-t)x}\right]_{\alpha}^{\infty}$$

$$= \frac{+\lambda e^{\lambda\alpha}}{\lambda-t} \cdot e^{-\lambda\alpha} \cdot e^{t\alpha}$$

$$= \frac{\lambda e^{t\alpha}}{\lambda-t} \quad \text{provided } t < \lambda$$

ii) $M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} e^{t\alpha}$

$$M_X'(t) = \frac{1}{\lambda} \cdot \left(1 - \frac{t}{\lambda}\right)^{-2} \cdot e^{t\alpha} + \alpha\left(1 - \frac{t}{\lambda}\right)^{-1} \cdot e^{t\alpha}$$

$$E(X) = M_X'(0) = \frac{1}{\lambda} + \alpha$$

$$M''_X(t) = \frac{2}{\lambda^2}\left(1-\frac{t}{\lambda}\right)^{-3}\cdot e^{t\alpha} + \frac{2\alpha}{\lambda}\left(1-\frac{t}{\lambda}\right)^{-2}\cdot e^{t\alpha}$$
$$+ \alpha^2\left(1-\frac{t}{\lambda}\right)^{-1}\cdot e^{t\alpha}$$

$$E(X^2) = M''_X(0) = \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2$$

$$V(X) = \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2 - \left(\frac{1}{\lambda} + \alpha\right)^2$$

$$= \frac{1}{\lambda^2}$$

5) $f(x,y) = ax(x+y)$, $0<x<5$,
$10<y<20$

$$f_y(y) = \int_0^y ax(x+y)\cdot dx = \int_0^y ax^2 + axy \cdot dx$$

$$= \left[\frac{ax^3}{3} + \frac{ax^2y}{2}\right]_0^y = \frac{ay^3}{3} + \frac{ay^3}{2}$$

$$= \frac{5ay^3}{6}$$

$$E(Y/X) = \int_{10}^{20} \frac{y\cdot f_{xy}(x,y)}{f_x(x)} \cdot dy$$

$$= \frac{6}{5}\int_{10}^{20} \frac{\cancel{y}\cdot \cancel{a}x(x+y)}{\cancel{a}y^{\cancel{3}2}} \cdot dy$$

$$= \frac{6}{5}\int_{10}^{20} \frac{x^2 + xy}{y^2}\, dy$$

$$= \frac{6}{5}\int_{10}^{20} x^2y^{-2} + xy^{-1} \cdot dy$$

$$= \frac{6}{5}\left[\frac{x^2}{-2y} + x\log y\right]_{10}^{20}$$

$$= \frac{6}{5}\left[\frac{x^2}{-40} + x\log 20 + \frac{x^2}{20} - x\log 10\right]$$

$$\Rightarrow \frac{6}{5}x\log 2 + \frac{6}{5}\times\frac{x^2}{40}$$

$$\Rightarrow \frac{6x}{5}\log 2 - \frac{3x^2}{100}$$

6) $X \sim Poi(\lambda)$, $Y \sim Poi(\mu)$, $M_X(t) = e^{\lambda(e^t-1)}$

$M_Y(t) = e^{\mu(e^t-1)}$

$Z = X - Y$

$$M_Z(t) = E(e^{tZ}) = E(e^{t(X-Y)})$$

$$= E(e^{tX}) \cdot E(e^{-tY})$$

$$= M_X(t) \cdot M_Y(-t)$$

$$= e^{\lambda(e^t-1)} \cdot e^{\mu(e^{-t}-1)} \Rightarrow Z \not\sim Poi(\lambda-\mu)$$

$$\cancel{= e^{\lambda(e^t-1+e^{-t}-1)}} \quad X - Y \not\sim Poi(\lambda-\mu)$$

$Z = X + Y$

$$M_Z(t) = E(e^{tZ}) = E(e^{t(X+Y)})$$

$$= E(e^{tX}) \cdot E(e^{tY})$$

$$= M_X(t) \cdot M_Y(t)$$

$$= e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)}$$

$$= e^{\lambda+\mu(e^t-1)}$$

$Z \sim Poi(\lambda+\mu)$

$\Rightarrow X + Y \sim Poi(\lambda+\mu)$

7)

		X		
	1	2	3	4
Y 1	0.2	0	0.05	0.15
2	0	0.3	0.1	0.2

i) $Var(X/Y=2)$

$$E(X^{2}/Y=2) = \frac{\sum x^{2} \cdot P(X=x, Y=2)}{P(Y=2)}$$

$$P(Y=2) = 0.6$$

$$E(X/Y=2) = 1 \times \frac{0}{0.6} + 2 \times \frac{0.3}{0.6} + 3 \times \frac{0.1}{0.6} + 4 \times \frac{0.2}{0.6}$$

$$= 0 + 1 + \cancel{0.5} + \frac{1}{2} + \cancel{\tfrac{1}{4}}\ \frac{4}{3}$$

$$= \frac{6+3+8}{6} = \frac{17}{6}$$

$$E(X^{2}/Y=2) = 1 \times \frac{0}{0.6} + 4 \times \frac{0.3}{0.6} + 9 \times \frac{0.1}{0.6} + 16 \times \frac{0.2}{0.6}$$

$$= \frac{1.2 + 0.9 + 3.2}{0.6} = \frac{5.3}{0.6} = \frac{53}{6}$$

$$V(X/Y=2) = E(X^2/Y=2) - [E(X/Y=2)]^2$$
$$= \frac{53}{6} - \left(\frac{17}{6}\right)^2$$
$$= \frac{29}{36}$$

ii) $f(u,v) = \frac{48}{67}(2uv - u^2)$, $0 < u < 1$, $\frac{u}{2} < v < 2$

$E(U/V=v)$

$$f_V(v) = \frac{48}{67}\int_0^1 (2uv - u^2)\cdot du = \frac{48}{67}\left[u^2 v - \frac{u^3}{3}\right]_0^1$$
$$= \frac{48}{67}\left[v - \frac{1}{3}\right] = \frac{\cancel{48}16}{67}\left[3v - 1\right]$$

$$E(U/V=v) = \int_0^1 \frac{u\cdot f_{U,V}(u,v)\cdot du}{f_V(v)}$$
$$= \frac{48}{67}\int_0^1 \frac{u\cdot(2uv - u^2)\cdot du}{(3v-1)\frac{16}{67}}$$
$$= 3\left[\frac{\frac{2u^3 v}{3} - \frac{u^4}{4}}{3v - 1}\right]_0^1 = 3\left[\frac{\frac{2v}{3} - \frac{1}{4}}{3v - 1}\right]$$
$$= \cancel{3}\left[\frac{8v - 3}{4\cancel{12}(3v-1)}\right] = \frac{8v-3}{12v-4}$$

8) $X \sim Gamma(3,2)$, $E(X) = \frac{3}{2}$, $V(X) = \frac{3}{4}$

$E(Y/X=x) = 3x+1$, $V(Y/X=x) = 2x^2+5$

$$E(Y) = E(E(Y/X)) = E(3x+1)$$
$$= 3E(X)+1$$
$$= 3 \times \frac{3}{2} + 1$$
$$= 13/4$$

$$V(Y) = E(V(Y/X)) + V(E(Y/X))$$
$$= E(2x^2+5) + V(3x+1)$$
$$= 2E(X^2) + 5 + 9V(X)$$

$\left[\begin{array}{l} V(X) = E(X^2) - [E(X)]^2 \\ E(X^2) = \frac{3}{4} + \frac{9}{4} = 3 \end{array}\right.$

$$= 2 \times 3 + 5 + 9 \times \frac{3}{4}$$
$$= 11 + \frac{27}{4} = \frac{71}{4}$$

9) $Corr(X,Y) = -0.3 = \dfrac{Cov(X,Y)}{\sqrt{V(X) \cdot V(Y)}}$

$E(X) = 3, \quad V(X) = 4, \quad E(Y) = 4, \quad V(Y) = 1$

$Cov(X,Y) = -0.3 \times \sqrt{4 \times 1} = -0.6$

$Z = X + Y$

i) $Cov(X,Z) = Cov(X, X+Y)$

$= Cov(X,X) + Cov(X,Y)$

$= V(X) + (-0.6)$

$= 4 - 0.6 = 3.4$

ii) $V(Z) = Cov(Z,Z) = Cov(X+Y, X+Y)$

$= Cov(X,X) + 2Cov(X,Y) + Cov(Y,Y)$

$= V(X) + 2Cov(X,Y) + V(Y)$

$= 4 + 2 \times (-0.6) + 1$

$= 3.8$

10) $f_{XY}(x,y) = k x^{-\alpha} \cdot e^{-y/B} \quad , \quad \begin{matrix} 1 < x < \infty \\ 1 < y < \infty \end{matrix}$

$$\Rightarrow \int_{y=1}^{\infty} \int_{x=1}^{\infty} k x^{-\alpha} \cdot \cancel{y} e^{-y/B} \cdot dx \cdot dy = 1$$

$$\Rightarrow \int_{1}^{\infty} k \cancel{y} e^{-y/B} \left[\frac{x^{-\alpha+1}}{-\alpha+1} \right]_1^{\infty} \cdot dy = 1$$

$$\Rightarrow \int_{1}^{\infty} \frac{k e^{-y/B}}{\alpha - 1} \cdot dy = 1$$

$$\Rightarrow \frac{k}{\alpha - 1} \left[\frac{e^{-y/B}}{-1/B} \right]_1^{\infty} = 1$$

$$\Rightarrow \frac{-kB}{\alpha - 1} \left[-e^{-1/B} \right] = 1$$

$$\Rightarrow \frac{kBe^{-1/B}}{\alpha - 1} = 1$$

$$k \Rightarrow \frac{(\alpha - 1) \cdot e^{1/B}}{B}$$