Probability and stats

1. Let

$X \rightarrow$ be the number of home insurancy
$Y \rightarrow$ be the number of car insurence

$$X \sim N(800, 100^2)$$
$$Y \sim N(1200, 300^2)$$

Now,

$$P[(Y_1 + Y_2 + Y_3) > (X_1 + X_2 + X_3 + X_4) + 800]$$

$$= P[(Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) > 800]$$

Hence, $N(3 \times 1200 - 4 \times 800,\ 3 \times 300^2 + 4 \times 100^2)$

$$\sim N(400, 310000)$$

$$= P\left[Z > \frac{(800-400)}{\sqrt{310000}}\right]$$

$$= P(Z > 0.718)$$

$$= 1 - P(Z < 0.718) = \underline{0.236}$$

3.

(i) $f(x) = \lambda e^{-\lambda(x-\alpha)}$, $x \geq \alpha$

using the definition of MGF:

$$M_x(t) = E(e^{tx}) = \int_{\alpha}^{\infty} e^{tx} \lambda e^{-\lambda(x-\alpha)} dx$$

$$= \left[\frac{-\lambda e^{\lambda\alpha}}{\lambda - t} e^{-(\lambda - t)\alpha} \right]_{\alpha}^{\infty}$$

$$= \frac{-\lambda}{\lambda - t} e^{t\alpha}$$ provd $t < \lambda$

(ii) $M_x(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} e^{t\alpha}$

$$M'_x(t) = \frac{1}{\lambda}\left(1 - \frac{t}{\lambda}\right)^{-2} e^{tx} + \alpha\left(1 - \frac{t}{\lambda}\right)^{-1} e^{tx}$$

$$\Rightarrow E(x) = M'_x(0) = \frac{1}{\lambda} + \alpha$$

$$M''_x(t) = \frac{2}{\lambda^2}\left(1 - \frac{t}{\lambda}\right)^{-3} e^{tx} + \frac{2\alpha}{\lambda}\left(1 - \frac{t}{\lambda}\right)$$
$$+ \alpha^2 \left(1 - \frac{t}{\lambda}\right)^{-1} e^{tx}$$

$$\Rightarrow E(x^2) = M''_x(0) = \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2$$

$$Var(x) = \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2 - \left(\frac{1}{\lambda} + \alpha\right)$$

$$Var(x) = \frac{1}{\lambda^2}$$

classmate

Date
Page

4. Given that

$G_V(t) = \left(\frac{p}{1-qt}\right)^k$ and $p+q = 1$

$P(X=n) = \frac{1}{n!} G_{X}^{(n)}(0)$

$P(V=4) = \frac{1}{4!} G_V^{4}(0)$

$G'_V(t) = kp^k q (1-qx)^{-k-1}$

$G''_V(t) = -(-k-1)kp^k q^2 (1-qx)^{-k-2}$

$G'''_V(t) = (-k-2)(-k-1)kp^2 q^3 (1-qx)^{-k-3}$

$G''''_V(t) = -(k-3)(-k-2)(-k-1)kp^k q^4 (1-q^x)$

$G^4(0) = -(-k-3)(-k-2)(-k-1)kp^x q^4$

5. Given that

$f(x,y) = ax(x+y)$ $\quad 0 < x < 5$
$10 < y < 20$

To calculate $P(Y > \frac{1}{3}X + 10)$ & $E(Y|X)$

$f\cancel{(x,y)} = \int_{10}^{20}\int_{0}^{5} ax(x+y)\,dx\,dy = 1$

$= \int_{10}^{20}\int_{0}^{5} a(x^2+xy)\,dx\,dy = 1$

$$= \int_{10}^{20} a\left[\frac{x^3}{3} + \frac{x^2}{2}y\right]_0^5 dy = 1$$

$$= \int_{10}^{20} a\left[\frac{125}{3} + \frac{25}{2}y\right] dy = 1$$

$$= a\left[\frac{125}{3}y + \frac{25}{4}y^2\right]_{10}^{20} = 1$$

$$= a\left[\frac{2500}{3} + 2500 - \frac{1250}{3} - 625\right] = 1$$

$$= a(2291.66) = 1$$

$$a = \underline{0.00044}$$

$$E(Y/X) = \int_y y \cdot f\left(\frac{x,y}{f(x)}\right) dy$$

$$f(x) = \int_y f(x,y)\, dy$$

$$= \int_{10}^{20} a(x + xy)\, dy$$

$$= a\left[x^2 y + \frac{xy^2}{2}\right]_{10}^{20}$$

$$= a\left[20x^2 + 200x - 10x^2 - 50x\right]$$

$$= a\left[10x^2 + 150x\right]$$

classmate

Date

Page

$$E(Y|x) \int_{10}^{20} y \frac{\cancel{x}}{x}\left[\frac{x^2+xy}{10x^2+150x}\right] dy$$

$$= \int_{10}^{20} y \frac{(x+y)}{(10x+150)} dy$$

$$= \frac{1}{10x+150}\left[\frac{y^2}{2}x+\frac{y^3}{3}\right]_{20}^{20}$$

$$= \frac{1}{10x+150}\left(150x+\frac{7000}{3}\right)$$

6. Given that

$X \sim Poi(\lambda)$

$Y \sim Poi(\mu)$

X has MGF

$$M_x(t) = e^{\lambda(e^t-1)}$$

Y has MGF

$$M_y(t) = e^{\lambda(e^t-1)}$$

$$M_x(t)\ M_y(t) = e^{\lambda(e^t-1)} e^{-\mu(e^t-1)}$$

$$= e^{(\lambda-\mu)(e^t-1)}$$

it is moment generating function of a poisson with $\lambda-\mu$

classmate

Date ____

Page ____

$\Rightarrow X - Y \sim Poi(\lambda - \mu)$

So the statement is true

$X \sim Poi(\lambda)$ & $Y \sim Poi(\mu)$

$M_x(t) = e^{\lambda(e^t - 1)}$

$M_y(t) = e^{\mu(e^t - 1)}$

So $X+Y$ has MGF

$M_x(t)\ M_y(t) = e^{\lambda(e^t-1)} e^{\mu(e^t-1)}$
$= e^{(\lambda+\mu)(e^t-1)}$

$\Rightarrow$ MGF of poisson $(\lambda+\mu)$

So $X+Y \sim Poi(\lambda+\mu)$

The other statement is also true.

ii) $var\left(\frac{X}{Y} = 2\right)$ $\quad E\left(\frac{X^2}{Y}\right) = 2 - E^2\left(\frac{X}{Y} = 2\right)$

$E(X|Y) = 2 \cdot \frac{0}{0.6} \times 1 + \frac{2 \times 0.2}{0.6} + \frac{3 \times 0.1}{0.6} + \frac{4 \times 0.2}{0.6}$

$= \frac{17}{6}$

$$E\left(x^2 / y=2\right) = 1^2 \times \frac{0.}{0.6} + 2^2 \times \frac{0.3}{0.6} + 3^2 \times \frac{0.1}{0.6} + 4$$

$$= \frac{53}{6}$$

So $\operatorname{var}\left[\frac{x}{4} = 2\right] = \frac{53}{6} - \left(\frac{17}{6}\right)^2$

$$= \frac{29}{36} = \underline{\underline{0.80556}}$$

ii) Now the conditional expectation

$$E(u / v = v) = \int_u u f(u|v)\, du$$

$$f(v) = \int_0^1 \frac{48}{67}\left(2uv - u^2\right) du$$

$$= \frac{48}{67}\left[u^2 v - \frac{u^3}{3}\right]_0^1$$

$$= \frac{48}{67}\left[v - \frac{1}{3}\right]$$

$$\Rightarrow f(u|v) = \frac{f(u,v)}{f(v)} = \frac{\frac{48}{67}\left(2uv - u^2\right)}{\frac{48}{67}\left(v - \frac{1}{3}\right)}$$

$$= \left(\frac{2uv - u^2}{v - \frac{1}{3}}\right)$$

$$\text{So, } E(U|V=v) = \int_0^1 \frac{2u^2 v - u^3}{v - \frac{1}{3}}\, du$$

$$= \left[\frac{\frac{2}{3}u^3 v - \frac{u^4}{4}}{v - \frac{1}{3}}\right]_0^1$$

$$= \frac{\frac{2}{3}v - \frac{1}{4}}{v - \frac{1}{3}}$$

8. Now,

$X \sim \text{Gamma}(3, 2)$

$E[Y|X=x] = 3x+1$

$\text{Var}(Y|X=x) = 2x+5$

$E(Y) = E[E(Y|X)] = E(3x+1)$

$= 3E(x)+1$

As $E(x) = \frac{3}{2} = 1.5$

$\Rightarrow E(Y) = 3(1.5)+1$

$= \underline{5.5}$

classmate

Date
Page

Here,

$var(Y) = E[var(Y|X)] + var[E(Y|X)]$

$= var(Y) = E[2x^2+5] + var[3x+1]$

$= 2E[x^2] + 5 + 9\, var(x)$

$= 2\left[(1.5)^2 + \left(\frac{9}{16}\right)\right] + 9\left(\frac{3}{4}\right)$

$= 2(2.3125) + 6.75$

$var(Y) = \underline{\underline{12.375}}$

9\. Given that,

$E(x) = 3$

$SD(x) = 2$

$E(Y = 4$

$SD(Y) = 1$

correlation coefficient $= -0.3$

$Z = X + Y$

(i) $cov(X, Z) = cov(X, X+Y)$

$= cov(X, X) + cov(X, Y)$

$= var(X) + cov(X, Y)$

Now $cov(x, y) = -0.3 = \frac{cov(X, Y)}{\sqrt{var(X)\, var(Y)}} = \frac{cov(X}{\sqrt{4}}$

$cov(X, Y) = \underline{\underline{-0.6}}$

$\Rightarrow cov(X, Z) = 4 - 0.6 = \underline{\underline{3.4}}$

classmate

Date ______
Page ______

(ii) $var(z) = cov(z, z)$

$var(z) = cov(x+y, x+y)$

$= cov(x, x) + 2cov(x, y) + cov(y, y)$

$= var(x) + 2cov(x, y) + var(y)$

$= 4 + 2(-0.6) + 1$

$= 3.3$

10. $kx^{\alpha} e^{y/\beta}$

where $1 < x < \infty$

$1 < y < \infty$

So the PDF will integrate to 1

$$\int_y^{\infty}\int_x^{\infty} k x^{-\alpha} e^{y/\beta} \, dx \, dy = 1$$

Integrating over x,

$$\int_y^{\infty} k x^{\alpha} e^{-y/\beta} = ke^{-y/\beta}\left[\frac{x^{\alpha+1}}{\alpha+1}\right]_1^{\infty}$$

$$= \frac{ke^{-y/\beta}}{\alpha-1}$$

integrating over y

$$\int_1^\infty \frac{k e^{-y/\beta}}{\alpha-1} = \frac{k}{\alpha-1}\left[-\beta e^{-y/\beta}\right]_1^\infty$$

$$= \frac{k\beta e^{-1/\beta}}{\alpha-1}$$

Now putting it equal to 1

$$\frac{k\beta e^{-1/\beta}}{\alpha} = 1$$

$$K = \frac{(\alpha-1)e^{1/\beta}}{\beta}$$