

PROBABILITY & STATISTICS - ASSIGNMENT 2

- (1) Let X_i be claim size on home insurance.
Let Y_i be claim size on car insurance.

$$X \sim N(800, 100^2)$$

$$Y \sim N(1200, 300^2)$$

$$P(Y_1 + Y_2 + Y_3 > X_1 + X_2 + X_3 + X_4 + 800)$$

$$P[Y_1 + Y_2 + Y_3 - (X_1 + X_2 + X_3 + X_4) > 800]$$

$$(Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) \sim N(3 \times 1200 - 4 \times 800, 3 \times 300^2 + 4 \times 100^2) \sim N(100, 310000)$$

$$P(Z > \frac{800 - 100}{\sqrt{310000}})$$

$$\begin{aligned} P(Z > 0.71842) &= 1 - P(Z < 0.71842) \\ &= 1 - P(Z < 0.72) \\ &= 1 - 0.76424 \\ &= 0.23576 \end{aligned}$$

- (2) $N \sim \text{Negative Binomial}(K, p)$
 $P(N=n) = \binom{n+2}{n} (0.9)^3 (0.1)^n$

so, $K=3$ and $p=0.9$

$$M_Y(t) = M_N(\log M_X(t))$$

$$X \sim \text{Gamma}(6, 2)$$

$Y \rightarrow$ Total claim amount.

- (a) MGF of Y .

$$M_Y(t) = E[E e^{yt} | N=n]$$

$$\begin{aligned} E[e^{yt} | N=n] &= E(e^{X_1 t + X_2 t + \dots + X_n t}) \\ &= E(e^{tX_1}) \cdot E(e^{tX_2}) \cdot \dots \cdot E(e^{tX_n}) \\ &= \left(1 - \frac{t}{2}\right)^{-6n} \end{aligned}$$

$$\begin{aligned} \text{so, } E[E e^{yt} | N=n] &= E\left[\left(1 - \frac{t}{2}\right)^{-6n}\right] \\ &= E\left[e^{n \log \left(1 - \frac{t}{2}\right)^{-6}}\right] \\ &= M_N\left[\log \left(1 - \frac{t}{2}\right)^{-6}\right] \end{aligned}$$

$$M_N(t) = \left(\frac{pe^t}{1-qe^t} \right)^K = \left[\frac{pe^{\log(1-t/2)-6}}{1-qe^{\log(1-t/2)-6}} \right]^K$$

$$\begin{aligned} \textcircled{99} \quad V(Y) &= E(N) V(X) + [E(X)]^2 \cdot V(N) \\ &= \left[\frac{3}{0.9} \times \frac{6}{4} \right] + \left(\frac{6}{2} \right)^2 \times \frac{(3)(0.1)}{(0.9)^2} \end{aligned}$$

Standard ~~dev~~ Deviation = 2.887

$$\begin{aligned} \textcircled{39} \quad f(x) &= \lambda e^{-\lambda(x-\alpha)} \\ \text{MGF} &= E(e^{tx}) \\ &= \int_{\alpha}^{\infty} e^{tx} \lambda e^{-\lambda(x-\alpha)} dx \end{aligned}$$

$$\begin{aligned} M_{\lambda t} &= \lambda \int_0^{\infty} e^{tx} e^{-\lambda x} e^{\lambda x} dx \\ &= e^{\lambda x} \lambda \int_0^{\infty} e^{-(\lambda-t)x} dx \\ &= \frac{e^{\lambda x} \lambda}{t - \lambda} \left[e^{-(\lambda-t)x} \right]_0^{\infty} \\ &= \frac{e^{\lambda x} \lambda}{t - \lambda} \left[-e^{-(\lambda-t)x} \right] \\ &= \frac{e^{\lambda x} \lambda}{\lambda - t} e^{-\lambda x} e^{tx} \\ &= e^{tx} \frac{\lambda}{\lambda - t} \end{aligned}$$

$$\textcircled{90} \quad M_x(t) = \frac{\lambda e^{tx}}{\lambda - t} = \lambda e^{tx} (\lambda - t)^{-1}$$

$$\begin{aligned} M_{n^2}(t) &= \lambda [e^{tx}(\lambda-t)^{-2} + (\lambda-t)^{-1}e^{tx}(x)] \\ &= \lambda [(\lambda-t)^{-1}e^{tx}(x) + e^{tx}(\lambda-t)^{-2}] \\ &= \lambda e^{tx}(\lambda-t)^{-2} [x(\lambda-t) + 1] \end{aligned}$$

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$$\begin{aligned} \boxed{t=0} &= \lambda(\lambda)^{-2} [\alpha(\lambda) + 1] \\ &= \frac{1}{\lambda} [\lambda\alpha + 1] \\ &= \frac{\lambda\alpha + 1}{\lambda} \end{aligned}$$

$$M_x''(t) = \lambda [\alpha e^{t\alpha} (\lambda - t)^{-2} + \alpha^2 (\lambda - t)^{-1} e^{t\alpha} + e^{t\alpha} (t+2)(\lambda - t)^{-3} + (\lambda - t)^{-2} e^{t\alpha}]$$

Put $t=0$

$$\begin{aligned} &\lambda [\alpha(\lambda)^{-2} + \alpha^2(\lambda)^{-1} + (2)(\lambda)^{-3} + \alpha(\lambda)^{-2}] \\ &\lambda \left[\frac{2\alpha}{\lambda^2} + \frac{\alpha^2}{\lambda} + \frac{2}{\lambda^3} \right] \end{aligned}$$

$$E(x^2) = \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2}$$

$$\begin{aligned} V(x) &= E(x^2) - [E(x)]^2 \\ &= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left[\alpha + \frac{1}{\lambda} \right]^2 \end{aligned}$$

$$\begin{aligned} 0 &= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left(\alpha^2 + \frac{1}{\lambda^2} + \frac{2\alpha}{\lambda} \right) \\ &= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \alpha^2 - \frac{1}{\lambda^2} - \frac{2\alpha}{\lambda} \\ &= \frac{1}{\lambda^2} \end{aligned}$$

$$\textcircled{4} G_v(t) = \frac{(p)^k}{(1-qt)}$$

$$G_v(t) = \sum t^v \times P(v=v)$$

$$P(v=4) = \frac{(k+4)}{15} p^k q^k$$

$$= \frac{(k+4)}{4! \cdot 15} p^k q^k$$

$$= \frac{(k+3)!}{(k-1)! \cdot 4!} p^k (1-p)^k$$

5) $f(x, y) = ax(x+y)$
 P $\left(y > \frac{1}{3}x + 10\right)$

$$a \int_{10}^{20} \int_0^5 (x^2 + xy) dx dy = 1$$

$$\int_{10}^{20} \left[\frac{x^3}{3} + \frac{x^2 y}{2} \right]_0^5 dy = \frac{1}{a}$$

$$\int_{10}^{20} \left(\frac{125}{3} + \frac{25}{2} y \right) dy = \frac{1}{a}$$

$$\left[\frac{125}{3} y + \frac{25}{4} y^2 \right]_{10}^{20} = \frac{1}{a}$$

$$833.33 + 2500 - (416.667 + 625) = \frac{1}{a}$$

$$a = 0.00043636.$$

P $\left(y > \frac{1}{3}x + 10\right)$

$$f(y) = \frac{3}{6875} \int_0^5 (x^2 + xy) dx$$

$$= \frac{3}{6875} \left[\frac{x^3}{3} + \frac{x^2 y}{2} \right]_0^5$$

$$= \frac{3}{6875} \left[\frac{125}{3} + \frac{25y}{2} \right]$$

$$\text{So, } \frac{3}{6875} \int_{\frac{1}{3}x+10}^{20} \left(\frac{125}{3} + \frac{25y}{2} \right) dy$$

$$= \frac{3}{6875} \left[\frac{125y}{3} + \frac{25y^2}{4} \right]_{\frac{1}{3}x+10}^{20}$$

$$= \frac{3}{6875} \left[\frac{125}{3} (20) + \frac{25}{4} (400) - \frac{125}{3} \left(\frac{1}{3}x + 10 \right) - \frac{25}{4} \left(\frac{1}{3}x + 10 \right)^2 \right]$$

$$= \frac{3}{6875} \left[\frac{2500}{3} + 2500 - \frac{125x}{9} - \frac{1250}{3} - \frac{25}{4} \left[\frac{1}{9}x^2 + 100 + \frac{2x}{3} \right] \right]$$

$$= \frac{3}{6875} \left[\frac{2500}{3} + 2500 - \frac{125x}{9} - \frac{1250}{3} - \frac{25x^2}{36} - \frac{625}{2} - \frac{25x}{6} \right]$$

999 $E(Y/X)$

$$f_{Y/X} = \frac{f(x,y)}{f(x)}$$

$$f(x) = \frac{3}{6875} \int_{10}^{20} x^2 + xy \, dy$$

$$= \frac{3}{6875} \left[x^2 y + \frac{xy^2}{2} \right]_{10}^{20}$$

$$= \frac{3}{6875} [20x^2 + x(200) - 10x^2 - 50x]$$

$$= \frac{3}{6875} [10x^2 + 150x]$$

$$= \frac{30x}{6875} [x + 15]$$

$$f_{Y/X} = \frac{3}{6875} \frac{x(x+y)}{30x} \times 6875(x+15)$$

$$\text{so, } E(Y/X) = \int_{10}^{20} y \cdot \frac{x+y}{10(x+15)} \, dy$$

$$= \frac{1}{10(x+15)} \int_{10}^{20} (xy + y^2) \, dy$$

$$= \frac{1}{10(x+15)} \left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_{10}^{20}$$

$$= \frac{1}{10(x+15)} \left[200x + \frac{8000}{3} - 50x - \frac{1000}{3} \right]$$

$$= \frac{1}{10(x+15)} \left[150x + \frac{7000}{3} \right]$$

$$= \frac{1}{x+15} \left[15x + \frac{700}{3} \right]$$

6

$$X \sim \text{Poi}(\lambda)$$

$$M_X(t) = e^{\lambda(e^t - 1)}$$

$$Y \sim \text{Poi}(\alpha)$$

$$M_Y(t) = e^{\alpha(e^t - 1)}$$

$$\text{so, } Z = X - Y$$

$$M_Z(t) = E(e^{tz}) = E(e^{t(X-Y)})$$

$$\begin{aligned}
 &= E(e^{tx} \cdot e^{-ty}) \\
 &= E(e^{tx}) \times E(e^{-ty}) \\
 &= \frac{E(e^{tx})}{E(e^{-ty})}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}} \\
 &= e^{(\lambda-\mu)(e^t-1)}
 \end{aligned}$$

MGF of $Z = e^{(\lambda-\mu)(e^t-1)} \sim \text{Pois}(\lambda+\mu)$
 Let $Z = X + Y$

$$\begin{aligned}
 E(e^{tz}) &= e^{\lambda(e^t-1)} \times e^{\mu(e^t-1)} \\
 &= e^{(\lambda+\mu)(e^t-1)} \\
 Z &\sim \text{Pois}(\lambda+\mu)
 \end{aligned}$$

(7)

		X			
		1	2	3	4
Y	1	0.2	0	0.05	0.15
	2	0	0.3	0.1	0.2

$$\begin{aligned}
 \textcircled{1} V(X/Y=2) &= E(X^2|Y=2) - [E(X|Y=2)]^2 \\
 E[X^2|Y=2] &= \frac{\sum x^2 P(X=x, Y=2)}{P(Y=2)}
 \end{aligned}$$

where $x=1$, $x=2$, $x=3$, $x=4$

$$\begin{aligned}
 &= \frac{(1)^2 P(X=1, Y=2)}{P(Y=2)} + \frac{(2)^2 P(X=2, Y=2)}{P(Y=2)} + \frac{(3)^2 P(X=3, Y=2)}{P(Y=2)} + \frac{(4)^2 P(X=4, Y=2)}{P(Y=2)} \\
 &= \frac{(1)(0)}{0.6} + \frac{(4)(0.3)}{0.6} + \frac{(9)(0.1)}{0.6} + \frac{(16)(0.2)}{0.6} \\
 &= 8.8333
 \end{aligned}$$

$$E[X/Y=2] = \frac{\sum x P(X=x, Y=2)}{P(Y=2)}$$

$$\frac{(1)(0)}{0.6} + \frac{(2)(0.3)}{0.6} + \frac{(3)(0.1)}{0.6} + \frac{(4)(0.2)}{0.6}$$

$$= \frac{0.6 + 0.3 + 0.8}{0.6}$$

$$= 2.8333$$

$$\therefore v(x/y=z) = 8.8333 - (2.8333)^2$$

$$= 0.807$$

$$(ii) f(u, v) = \frac{48}{67} (2uv - u^2)$$

$$\frac{f(u, v)}{f(v)} = f(u|v) = v$$

$$f(v) = \frac{48}{67} \int_0^v (2uv - u^2) du$$

$$= \frac{48}{67} \left[uv^2 - \frac{u^3}{3} \right]_0^v$$

$$= \frac{48}{67} \left[v^3 - \frac{1}{3} \right]$$

$$= \frac{16}{67} (3v - 1)$$

$$f(v) = \frac{16}{37} (3v - 1)$$

$$\frac{f(u, v)}{f(v)} = \frac{48}{67} \frac{(2uv^2 - u^2)}{16(3v - 1)} \times 67$$

$$= \frac{3}{3v - 1} (2uv^2 - u^2) \quad \text{BUT WE CAN}$$

$$f(u|v=v) = \frac{3}{3v - 1} \int_0^v v (2uv^2 - u^2) du$$

$$= \frac{3}{3v - 1} \left[\frac{2u^3 v^2}{3} - \frac{u^4}{4} \right]_0^v$$

$$= \frac{3}{3v - 1} \left[\frac{2v^2}{3} - \frac{1}{4} \right]$$

$$= \frac{3}{3v - 1} \left[\frac{8v^2 - 3}{12} \right] = \frac{8v^2 - 3}{4(3v - 1)}$$

8) $X \sim \text{Gamma}(3, 2)$

$$E(Y/X = x) = 3x + 1$$

$$\text{Var}(Y/X = x) = 2x^2 + 5$$

$$E(Y) = E[E(Y/X = x)]$$

$$= E(3X + 1)$$

$$= 3E(X) + 1$$

$$= 3\left(\frac{3}{2}\right) + 1$$

$$= \frac{9}{2} + 1$$

$$= \frac{9+2}{2}$$

$$= \frac{11}{2}$$

$$V(Y) = E[V(X/Y)] + V[E(X/Y)]$$

$$= E(2x^2 + 5) + V(3x + 1)$$

$$= 2E(X^2) + 5 + 9V(X)$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$\frac{3}{4} + \frac{9}{4} = [E(X)]^2$$

$$E(X^2) = \frac{12}{4}$$

$$E(X^2) = 3$$

$$V(Y) = 2(3) + 5 + 9\left(\frac{3}{4}\right)$$

$$= 11 + \frac{27}{4}$$

$$= \frac{44+27}{4}$$

$$= \frac{71}{4}$$

$$\begin{aligned} \textcircled{9} \quad E(X) &= 3 & V(X) &= 4 \\ E(Y) &= 4 & V(Y) &= 1 \\ \text{correlation coeff} &= -0.3\sqrt{4} = \text{cov}(X, Y) \\ &= -0.6 \end{aligned}$$

$$Z = X + Y$$

$$\begin{aligned} \textcircled{9} \quad \text{cov}(X, X+Y) &= \text{cov}(X, X) + \text{cov}(X, Y) \\ &= V(X) + \text{cov}(X, Y) \\ &= 4 + (-0.6) \\ &= 3.4 \end{aligned}$$

$$\begin{aligned} \textcircled{99} \quad \text{Var}(Z) &= \text{cov}(X+Y, X+Y) \\ &= \text{cov}(X, X) + 2\text{cov}(X, Y) + \text{cov}(Y, Y) \\ &= V(X) + 2\text{cov}(X, Y) + V(Y) \\ &= 4 + 2(-0.6) + 1 \\ &= 5 - 1.2 \\ &= 3.8 \end{aligned}$$

$$\textcircled{10} \quad f(x, y) = k x^{-\alpha} e^{-y/\beta}$$

$$k \int_1^{\infty} \int_1^{\infty} x^{-\alpha} e^{-y/\beta} dx dy = 1$$

$$k \int_1^{\infty} e^{-y/\beta} \left[\frac{x^{-(\alpha-1)}}{-\alpha+1} \right]_1^{\infty} dy = 1$$

$$\frac{k}{1-\alpha} \int_1^{\infty} e^{-y/\beta} [-1] dy$$

$$\frac{k}{\alpha-1} \int_1^{\infty} e^{-y/\beta} dy = 1$$

$$\frac{k}{\alpha-1} \left[\frac{e^{-y/\beta}}{(-1/\beta)} \right]_1^{\infty} - \frac{k}{\alpha-1} (\beta) (e^{-1/\beta}) = 1$$

$$k = \frac{\alpha-1}{(\beta)(e^{-1/\beta})}$$