Probability Statistics
Assignment - 2

1. Let $X \rightarrow$ home insurance claim size
$Y \rightarrow$ car insurance claim size

$X \sim N(800, 100)$
$Y \sim N(1200, 300)$

To find: ~~X~~ $P(Y > 800)$

Sol^n: $P(Y > 800) = P\left(\frac{Y-\mu}{\sigma} > \frac{800-1200}{300}\right)$

$= P\left(Z > -\frac{4}{3}\right)$

$= P\left(Z < \frac{4}{3}\right)$

$= P(Z < 1.33)$

$= 0.90824$

2. $N \rightarrow$ no. of claims
$N \sim$ NegBin such that

$P(N = n) = \binom{n+2}{n} 0.9^3 0.1^n$

Let $X \rightarrow$ claim amounts (in 1000's)
$X \sim$ Gamma $(6, 2)$

X_i's are iid's, X & N are independent
$Y \rightarrow$ Total Claim amount

(i) MGF of Y:
$Y = X_1 + X_2 + X_3 + X_4 \ldots + X_N$

$M_Y(t) = E(e^{tY})$
$= E(E(e^{tY}/N))$

$= E(E(e^{t(x_1+x_2+x_3+\dots+x_N)}/N))$

$= E(E(e^{t(x_1+x_2+x_3\dots x_n)}))$

$= E(E(e^{tx_1}) \cdot E(e^{tx_2}) \dots E(e^{tx_n}))$

$= E(M_{x_1}(t) M_{x_2}(t) \dots M_{x_n}(t))$

Since x_i are iid's

$= E[(M_x(t))^n]$

$= E[e^{\ln(M_x(t))^n}]$

$= E[e^{n\ln(M_x(t))}]$

Ans $\Rightarrow \boxed{M_Y(t) = M_N(\ln(M_x(t)))}$

ii Standard deviation of total claim amount:

$\sigma(S) = \sqrt{V(S)}$

Using EV-VE result

$V(S) = E(V(S/N)) + V(E(S/N))$

$= E(n(V(x))) + V(nE(x)) = E(N)V(x) + [E(x)]^2 V(N)$

$= \left(\frac{r}{p}\right)\left(\frac{\alpha}{\lambda^2}\right) + \left(\frac{\alpha^2}{\lambda^2}\right)\left(\frac{rq}{p^2}\right)$

$= \left(\frac{3}{0.9}\right)\left(\frac{\cancel{6}^{3}}{\cancel{4}_{2}}\right) + \left(\frac{\cancel{36}^{9}}{\cancel{4}}\right)\left(\frac{3\times 0.1}{(0.9)^2}\right)$

$= 5 + 3.33$

$= 8.33$

Ans $\boxed{\sigma(S) = 2.8867}$

Q.1 $X \sim Exp(\lambda)$

$$f(x) = \lambda e^{-\lambda(x-\alpha)} \quad , x \geq \alpha, \; \lambda, \alpha > 0$$

$$M_X(t) = \int_{\alpha}^{\infty} e^{tx} \lambda e^{-\lambda(x-\alpha)} dx$$

$$= \lambda \int_{\alpha}^{\infty} e^{tx} e^{-\lambda x} e^{\lambda \alpha} dx$$

$$= \lambda e^{\lambda \alpha} \int_{\alpha}^{\infty} e^{tx} e^{-\lambda x} dx$$

$$= \lambda e^{\lambda \alpha} \int_{\alpha}^{\infty} e^{(t-\lambda)x} dx$$

$$= \frac{\lambda e^{\lambda \alpha}}{t-\lambda} \left[e^{(t-\lambda)x} \right]_{\alpha}^{\infty}$$

$$= - \frac{\lambda e^{\lambda \alpha}}{t-\lambda} e^{(t-\lambda)\alpha}$$

$$= \frac{-\lambda e^{\lambda \alpha} e^{-\lambda \alpha} e^{t\alpha}}{t-\lambda}$$

$$\boxed{M_X(t) = \frac{-\lambda e^{t\alpha}}{t-\lambda}}$$

(ii) $E(X) = M'_X(t)$

$$= -\lambda \left[\frac{(t-\lambda)\alpha e^{t\alpha} - e^{t\alpha}}{(t-\lambda)^2} \right]$$

Sub $t = 0$

$$E(X) = -\lambda \left[\frac{-\lambda \alpha - 1}{\lambda^2} \right]$$

$$= \boxed{\frac{\lambda \alpha + 1}{\lambda}}$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= M''_x(t) - [M'_x(t)]^2$$

$$= \frac{d}{dt}\left[-\lambda\left(\frac{\alpha e^{t\alpha}}{(t-\lambda)} - \frac{e^{t\alpha}}{(t-\lambda)^2}\right)\right] - \left(\frac{\lambda\alpha+1}{\lambda}\right)^2$$

$$= -\lambda\left[\frac{(t-\lambda)[\alpha^2 e^{t\alpha}] - \alpha e^{t\alpha}}{(t-\lambda)^2} - \frac{(t-\lambda)^2\alpha e^{t\alpha} - e^{t\alpha}2(t-\lambda)}{(t-\lambda)^4}\right]$$

$$- \left(\frac{\lambda\alpha+1}{\lambda}\right)^2$$

$$=$$

Sub $t = 0$

$$\Rightarrow -\lambda\left[\left(\frac{-\lambda[\alpha^2] - \alpha}{\lambda^2}\right) - \left(\frac{\lambda^2\alpha + 2\lambda}{\lambda^4}\right)\right] - \left(\frac{\lambda\alpha+1}{\lambda}\right)^2$$

Ans: $\Rightarrow V(x) = \frac{\alpha^2\lambda + \alpha}{\lambda} + \frac{\lambda^2\alpha + 2\lambda}{\lambda^3} - \left(\frac{\lambda\alpha+1}{\lambda}\right)^2$

4. PGF : $G_V(t) = \left(\frac{p}{1-qt}\right)^k$; $p+q=1$

$\cancel{G_V(t) = p^k(1-qt)^{-k}}$

$\cancel{= p^k(1-(1-p)t)^{-k}}$

$\cancel{= p^k(pt)^{-k}}$

$\cancel{G_V(t) = \left(\frac{1}{t}\right)^k}$

$\cancel{\left(\frac{1}{t}\right)^k} =$

$$P(x=4) = \frac{1}{4!} G^{iv}_x(0)$$

$\cancel{\Rightarrow G'_x(0) =}$

$\rightarrow G'_x(t) = p^k(-k)(1-qt)^{-k-1}[-q]$

$\rightarrow G''_x(t) = kqp^k(-k-1)(1-qt)^{-k-2}[-q]$

$\Rightarrow G'''_x(t) = k(k+1)q^2p^k(-k-2)(1-qt)^{-k-3}(-q)$

$\Rightarrow G^{iv}_x(t) = k(k+1)(k+2)q^3p^k(-k-3)(1-qt)^{-k-4}(-q)$

$$\Rightarrow G''''_x(\theta) = k(k+1)(k+2)(k+3)q^4p^k$$

$$\rightarrow P(x=4) = \frac{1}{4!}\left[k(k+1)(k+2)(k+3)q^4p^k\right]$$

Ans $\boxed{P(x=4) = \frac{k(k+1)(k+2)(k+3)q^4p^k}{24}}$

5. $f(x,y) = ax(x+y)$; $0<x<5$
; $10<y<20$

$$\rightarrow \iint_{\text{all } x \text{ all } y} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_0^5\int_{10}^{20} ax(x+y)\,dy\,dx = 1$$

$$\Rightarrow \int_0^5 \left[ax^2y + \frac{ax}{2}y^2\right]_{10}^{20} dx = 1$$

$$\Rightarrow \int_0^5 (10ax^2 + 150ax)\,dx = 1$$

$$\Rightarrow \left[\frac{10}{3}ax^3 + 75ax^2\right]_0^5 = 1$$

$$\Rightarrow \frac{10\times125a}{3} + 75\times25a = 1$$

$$\Rightarrow 6875a = 3$$

$$\Rightarrow a = 0.000436$$

$$P\left(Y > \frac{1}{3}x + 10\right) = \int_{\cancel{10}}^{\cancel{50}} \int_{\frac{1}{3}x+10}^{20} \frac{3}{6875}(x^2 + xy)\,dx\,dy$$

$$= \cancel{\int_{10}^{50} \frac{3}{6875} \left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_{\frac{1}{3}x+10}^{20} dy}$$

$$\cancel{= \frac{3}{6875} \int_{10}^{20} \left[125 \right.}$$

$$= \int_0^5 \frac{3}{6875} \left[x^2 y + \frac{x y^2}{2}\right]_{\frac{1}{3}x+10}^{20} dx$$

$$= \frac{3}{6875} \int_0^5 x^2\left[10 - \frac{1}{3}x\right] + \frac{x}{2}\left[400 - \frac{1x^2}{9} - 100 - \frac{20x}{3}\right]$$

$$= \frac{3}{6875}\left[\frac{10x^3}{3} - \frac{1}{3}\frac{x^4}{4} + \frac{200x^2}{2} - \frac{1}{18}\left(\frac{x^4}{4}\right) - \frac{100x^2}{2\ 2} - \frac{20}{6}\frac{x^3}{3}\right]_0^5$$

$$= \frac{3}{6875}\left[\frac{40}{6}\left(\frac{x^3}{3}\right) - \frac{7}{18}\left(\frac{x^4}{4}\right) + 150\left(\frac{x^2}{2}\right)\right]_0^5$$

$$= \frac{3}{6875}\left[2092.0138\right]$$

Ans $= \underline{\underline{0.9128}}$

Scanned with CamScanner

$$E(Y/x) = \int_{10}^{20} y\, f_{Y/x}(x,y)\, dy$$

$$= \int_{10}^{20} y \frac{f(x,y)}{f_x(x)}\, dy$$

$$\cancel{= \frac{3}{6875} \int y(x^2+xy)}$$

$$\Rightarrow f_x(x) = \int_{10}^{20} a(x^2+xy)\, dy$$

$$= a\left[x^2 y + \frac{xy^2}{2}\right]_{10}^{20}$$

$$= a\left[10x^2 + 150x\right]$$

$$\Rightarrow E(Y/x) = \int_{10}^{20} \frac{y[a(x^2+xy)]}{a[10x^2+150x]}\, dy$$

$$= \frac{1}{10x^2+150x} \int_{10}^{20} x(yx+y^2)\, dy$$

$$= \frac{1}{10x+150} \left[\frac{y^2}{2}x + \frac{y^3}{3}\right]_{10}^{20}$$

Ans $$E(Y/x) = \boxed{\frac{150x + 2333.33}{10x+150}}$$

6. Acc. to Arthur's statement :

$x \sim Poi(\lambda)$
$y \sim Poi(\mu)$; such that
$x - y \sim Poi(\lambda - \mu)$; x & y are independent

let $z = x - z$
~~let~~ We know that :

$M_x(t) = e^{\lambda(e^t - 1)}$
$M_y(t) = e^{\mu(e^t - 1)}$.

→ MGF of z : $M_z(t) = E(e^{tz})$

$= E(e^{t(x-y)})$

$= E(e^{tx} \cdot e^{-ty})$

$= \dfrac{E(e^{tx})}{E(e^{ty})}$.

$= \dfrac{M_x(t)}{M_y(t)}$

$= \dfrac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}}$

$M_z(t) = e^{(\lambda-\mu)(e^t-1)}$

This is the MGF for a poisson distribution with parameter $(\lambda - \mu)$

$\Rightarrow z \sim Poi(\lambda - \mu)$

Acc to Christine's statement

$x + y \sim Poi(\lambda + \mu)$

let $A = x + y$

→ $M_A(t) = E(e^{tA})$

$= E(e^{t(x+y)})$

$= E(e^{tx} \cdot e^{ty})$

$= E(e^{tx}) \cdot E(e^{ty}) = M_x(t) \cdot M_y(t)$

$$M_A(t) = e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)}$$
$$= e^{(\lambda+\mu)(e^t-1)}$$

This the MGF for a poisson distri. with parameters $(\lambda+\mu)$

$\Rightarrow A \sim Poi(\lambda+\mu)$

7. (i) $V(X/_{Y=2}) = E(X^2/_{Y=2}) - [E(X/_Y)]^2$

$$E(X^2/_{Y=2}) = \sum_{\text{all } x} x^2 \frac{P(X=x, Y=2)}{P(Y=2)}$$

$$= \frac{(1\times 0) + (4\times 0.3) + (9\times 0.1) + (16\times 0.2)}{0+0.3+0.1+0.2}$$

$$= 8.833$$

$$E(X/_Y) = \sum_{\text{all } x} x \frac{P(X=x, Y=2)}{P(Y=2)}$$

$$= \frac{(1\times 0) + (2\times 0.3) + (3\times 0.1) + (4\times 0.2)}{0+0.3+0.1+0.2}$$

$$= 2.833$$

$\Rightarrow V(X/_{Y=2}) = 8.833 - (2.833)^2$
$$= 0.8052$$

ii. $f(U,V) = \frac{48}{67}(2UV - U^2)$; $0<U<1$
; $\frac{U}{2} < V < 2$

$$E(U/V=v) = \int_{\text{all } U} U f_{U/V}(U,V)\, dU$$

$$= \int_{\text{all } U} U \frac{f_{U,V}(U,V)}{f_V(v)}\, dU$$

$$f_V(v) = \int_0^1 \frac{48}{67}(2uv - u^2)\,du$$

$$= \frac{48}{67}\left[\frac{\cancel{2}u^2 v}{\cancel{2}} - \frac{u^3}{3}\right]_0^1$$

$$= \frac{48}{67}\left[v - 1\right]$$

$$E(U/V=v) = \int_0^1 \frac{u\left(\cancel{\frac{48}{67}}(2uv-u^2)\right)}{\cancel{\frac{48}{67}}(v-1)}\,du$$

$$= \frac{1}{v-1}\int_0^1 (2u^2 v - u^3)\,du$$

$$= \frac{1}{v-1}\left[\frac{2u^3 v}{3} - \frac{u^4}{4}\right]_0^1$$

$$= \frac{\frac{2}{3}v - \frac{1}{4}}{v-1}$$

Ans $$\boxed{E(U/V=v) = \frac{8v-3}{12}}$$

8. $X \sim \text{Gamma}(3,2)$; $\alpha = 3$, $\lambda = 2$

$E(Y/X=x) = 3x+1$; $V(Y/X=x) = 2x^2+5$

Mean:

$$E(Y) = E(E(Y/X))$$

$$= E(3x+1)$$

$$= 3E(X) + 1$$

$$= 3\left(\frac{\alpha}{\lambda}\right) + 1$$

$$= 3\left(\frac{3}{2}\right) + 1$$

$$E(Y) = \boxed{5.5}$$

$$V(Y) = E(V(Y/X)) + V(E(Y/X))$$

$$= E(2x^2+5) + V(3x+1)$$

$$= 2E(x^2) + 5 + 3V(x)$$

$$= 2\left[V(x) + [E(x)]^2\right] + 5 + 3V(x)$$

$$= 2\left[\frac{\alpha}{\lambda^2} + \frac{\alpha^2}{\lambda^2}\right] + 5 + 3\left(\frac{\alpha}{\lambda^2}\right)$$

$$= 2\left(\frac{3}{4}\right) + 2\left(\frac{9}{4}\right) + 5 + 3\left(\frac{3}{4}\right)$$

$$V(Y) = 13.25$$

Ans SD of $Y = \sqrt{V(Y)}$

$$= \underline{\underline{3.64}}$$

9. $X \rightarrow RV$ $\quad$ $Y \rightarrow RV$

$E(X) = 3$ $\quad$ $E(Y) = 4$

$\sigma_x = 2$ $\quad$ $\sigma_y = 1$

$\Rightarrow V(X) = 4$ $\quad$ $\Rightarrow V(Y) = 1$

$Corr(X,Y) = -0.3$

$Z = X + Y$

(i) $Cov(X,Z) = E(XZ) - E(X)E(Z)$

$$= E(X(X+Y)) - E(X)E(X+Y)$$

$$= E(X^2) + E(XY) - E(X)E(X)E(Y)$$

$$= [V(X) + (E(X))^2] + [(Corr(X,Y)\sigma_x\sigma_y) + E(X)E(Y)] - [E(X)]^2 E(Y)$$

$$= 4 + 3^2 + (-0.3)(2)(1) + (3)(4) - (3)^2(4)$$

Ans $Cov(X,Z) = \underline{\underline{-11.6}}$

Scanned with CamScanner

(ii) $V(z) = V(x+y)$

$= V(x) + V(y) + 2\,Cov(x,y)$

$= 4 + 1 + 2[(-0.3)(2)(1)]$

Ans $V(z) = 3.8$

10. $f(x,y) = kx^{-\alpha} e^{-\frac{y}{\beta}}$; $1 < x < \infty$

; $1 < y < \infty$

; $a > 1$, $\beta > 0$

$\Rightarrow \int_{\text{all } x} \int_{\text{all } y} f(x,y)\,dx\,dy = 1$

$\Rightarrow \int_{1}^{\infty} \int_{1}^{\infty} kx^{-\alpha} e^{-\frac{y}{\beta}}\,dy\,dx = 1$

$\Rightarrow \int_{1}^{\infty} k\,x^{-\alpha} (-\beta) \left[e^{-\frac{y}{\beta}}\right]_{1}^{\infty} dx = 1$

$\Rightarrow -\beta k \int_{1}^{\infty} x^{-\alpha} \left[e^{-\frac{1}{\beta}}\right] dx = 1$

$\Rightarrow e^{-\frac{1}{\beta}} \beta k \left[\frac{x^{-\alpha+1}}{1-\alpha}\right]_{1}^{\infty} = 1$

$\Rightarrow \frac{e^{-\frac{1}{\beta}} \beta k}{1-\alpha} \left[x^{-(\alpha-1)}\right]_{1}^{\infty} = 1$

$\Rightarrow \frac{k\beta e^{-\frac{1}{\beta}}}{1-\alpha} [-1] = 1$

Ans $\Rightarrow \boxed{k = \frac{\alpha - 1}{\beta e^{-\frac{1}{\beta}}}}$

Scanned with CamScanner