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Financial engineering
Assignment - 1

1

i) an arbitrage opportunity arises when you buy for cheap in one market and simultaneously sale for a higher price in another market.

ii) the law of one price state that in absence of friction between the markets globally. the price of an asset will be the same.
It is achieved by eliminating Price differences between markets.

iii) Strike Price of 3 month european Put = 120
Share price = 125
Price of the call option. = 30.
risk free rate = 5% p.a

(a) Price of Put by Put call Parity.

$P + S_0 = C + ke^{-rt}$

$P = 30 + 120 \times e^{-5\%/4} - 125$

$P = 23.51$

(b) If the Price of the Put is 23
an Investor or arbitrager ~~can~~ Short the Underlying ~~stock and take long Position~~
can take long Position in Put and Buy the underlying stock and simultaneously can ~~sell~~ Short the call and Borrow 118.51 Rs at the risk free rate of 5%.

to make an immediate Profit of. 0.51 Rs

Q4 Given: American call with 10 months to maturity ~~and risk~~
risk free rate = 9.5%

$K = Rs\ 350$

q = dividend = Rs. 10 $t = 3$ months ~~interval~~ and 6 month

On First dividend date that is after 3 months.

~~If we exercise the option. The payoff is~~

$\cancel{S_{t_1} - K - q_1}$

$\Rightarrow \cancel{S_{t_1} - 350 - 10}$

The Price of the stock drops on the dividend date so.
it is better not to exercise the ~~opt~~ call (American)
on the dividend date.

Let the dividend dates be $t_1 \leq t_2 < t_n$

Let the excercised Prior to ex dividend date then
the investor receives $S(t_n) - K$

The stock Price will drop to $S(t_n) - d$

Value of the option. $S(t_n) - d_n - Ke^{-r(T-t_n)}$

If $S(t_n) - d_n - Ke^{-r(T-t_n)} \geq \cancel{S(t_n)} \geq S(t_n) - K$ — ①

It is not optimal To exercise the option.

from ① $d_n \leq K(1 - e^{-r(T-t_n)})$

Puting values.

$350 \times (1 - e^{(-0.095 \cdot (0.8333 - 0.25))}) = 18.87$.

Hence it is not optimal to exercise it
on dividend date.

5. expected return = 16%
Volatility = 35%
S_0 = 254
K = 258
t = 6 months or 0.5 years

$S_t = S_0 e^{(\mu - \sigma^2/2)t + \sigma W_t}$

The Log normal distribution

$Ln(S_t) = Ln(S_0) + (\mu - \sigma^2/2)t$

Variance = $\sigma^2 t$.

$Ln(S_t)$ follows $\varphi(Ln 254 + (0.16 - \frac{(0.35)^2}{2})0.5, \sqrt{0.35 \times 0.5})$

$= \varphi(5.59, 0.247)$.

$P(X > 258) = 1 - P(X < 258) \Rightarrow 1 - N(5.55 - 5.59)/0.247$
$= 1 - N(-0.1364) = 0.5542$

~~The Put~~

Similarly,
The Put option will be exercised if stock price is below 258 in 6 months
and i.e $1 - 0.5542 = 0.4457$

2) Let $dx_t = A_t dt + B_t dz_t$

Where $A_t = \alpha\mu(T-t)$ $B_t = \sigma\sqrt{(T-t)}$. — (1)

$$dF = \frac{\partial f}{\partial x}\beta t\, dz_t + \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}A_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\beta t\right)dt \quad \text{(Ito's lemma)}$$

$$dF = -f\beta t\, dz_t + f\frac{\partial m}{\partial t}(T-t)dt - fA_t dt + \frac{1}{2}\beta t^2 dt$$

$\left[\text{Since } \frac{\partial f}{\partial x} = -e^{m(T-t)-x}\right]$ and $\left[\frac{df}{dx} = \frac{dm}{dt}(T-t)xe^{m(T-t)-x}\right.$ and $\left.\frac{d^2f}{dx^2} = e^{m(T-t)-x}\right]$

$$dF = f\left(\frac{dm}{dt}(T-t) - A_t + \frac{1}{2}\beta t^2\right)dt - f\beta t\, dz_t$$

For f to be a martingale, $\frac{\partial m}{\partial t}(T-t) - A_t + \frac{1}{2}\beta t^2 = 0$

Thus $\frac{dm}{dt}(T-t) = A_t - \frac{1}{2}\beta t^2$

Substituting Eq 1 above gives $\frac{dm}{dt} = \alpha\mu - \frac{1}{2}\sigma^2$.

3) Given

(1) $F(t,x) = e^{-t}x^2$

$$\frac{df}{dt} = -e^{-t}x^2 = -f$$

$$\frac{df}{dx} = 2e^{-t}x$$

$$\frac{d^2f}{dx^2} = 2e^{-t}$$

By Ito's Lemma

$$dy_t = \frac{df}{dt}dt + \frac{df}{dx}dx_t + \frac{1}{2}\frac{d^2f}{dx^2}\sigma^2 X_t^2\, dt$$

$$dy_t = -Y_t dt + 2e^{-t}X_t\, dX_t + e^{-t}\sigma^2 X_t^2\, dt$$

$$dy_t = -Y_t dt + 2e^{-t}X_t^2\, \frac{dX_t}{} + e^{-t}\sigma^2 X_t^2\, dt$$

$$-Y_t dt + 2Y_t[0.24dt\ \cancel{X}t + \sigma^2 Y_t dt]$$

$$\frac{dY_t}{Y_t} = \left[2\times(0.25) - 1 + \sigma^2\right]dt + 2\sigma\, dW_t$$

$$\frac{dY_t}{Y_t} = \left[\sigma^2 - 0.5\right] Y_t\, dt + 2\sigma\, Y_t\, dW_t$$

(ii) The process is martingale if drift is zero.
This means $\sigma^2 - 0.5 = 0$
i.e. $\sigma^2 = 0.5$.

(6)

(i) The given relationship can be written as:

$$S_t = S_0 e^{\mu t + \sigma B_t}$$

Since S_t is a function of standard Brownian motion, B_t, applying Ito's lemma, The SDE for the underlying stochastic process becomes

$$dB_t = 0 \times dt + 1 \times dB_t.$$

Let $G(t, B_t) = S_t = S_0 e^{\mu t + \sigma B_t}$; then

$$\frac{dG}{dt} = \mu S_0 e^{\mu t + \sigma B_t} = \mu S_t$$

$$\frac{dG}{dB_t} = \sigma S_0 e^{\mu t + \sigma B_t} = \sigma S_t$$

$$\frac{d^2 G}{dB_t^2} = \sigma^2 S_0 e^{\mu t + \sigma B_t} = \sigma^2 S_t.$$

Hence using Ito's Lemma ~~from Page 46 in~~

$\cancel{dG = [0 \times \sigma S_t + 1/2 \times 1^2 \times \sigma}$

$dS_t = (\mu + \frac{1}{2}\sigma^2) S_t dt + \sigma S_t dB_t$

Thus

$\frac{dS_t}{S_t} = \sigma dB_t + (\mu + \frac{1}{2}\sigma^2) dt.$

So $C_1 = \sigma$ and $C_2 = \mu + \frac{1}{2}\sigma^2.$

(ii) The expected value of S_t is

$E[S_t] = E[S_0 e^{\mu t + \sigma B_t}] = S_0 e^{\mu t} E[e^{\sigma B_t}]$

Since $B_t \sim N(0,t)$ its MGF is $E[e^{\theta B_t}] = e^{\frac{1}{2}\theta^2 t}.$

So, $E[S_t] = S_0 e^{\mu t} \times e^{\frac{1}{2}\sigma^2 t} = S_0 e^{\mu t + \frac{1}{2}\sigma^2 t}$

The variance of S_t is:

$Var[S_t] = E[S_t^2] - (E(S_t))^2 = E[S_0^2 e^{2\mu t + 2\sigma B_t}] - (S_0 e^{\mu t + \frac{1}{2}\sigma^2 t})^2$

$= S_0^2 e^{2\mu t} E[e^{2\sigma B_t}] - S_0^2 e^{2\mu t + \sigma^2 t} = S_0^2 e^{2\mu t + 2\sigma^2 t} - S_0^2 e^{2\mu t + \sigma^2 t}$

iii) $cov[S_{t_1}, S_{t_2}] = E[S_{t_1}, S_{t_2}] - E[S_{t_1}] E[S_{t_2}].$

from above

$E[S_{t_1}] = S_0 e^{\mu t + \frac{1}{2}\sigma^2 t_1}$ and $E[S_{t_2}] = S_0 e^{\mu t_2 + \frac{1}{2}\sigma^2 t_2}$

The expected value of the products is

$E[S_{t_1}; S_{t_2}] = E[S_0 e^{(\mu t_1 + \sigma B_{t_1})} S_0 e^{(\mu t_2 + \sigma B_{t_2})}]$

$= S_0^2 e^{\mu(t_1 + t_2)} E[e^{(\sigma B_{t_1} + \sigma B_{t_2})}]$

To evaluate this we need to split B_{t_2} into two independent components.

$B_{t_2} = B_{t_1} + (B_{t_2} - B_{t_1})$ where $B_{t_2} - B_{t_1} \sim N(0, t_2 - t_1).$

Hence

$E[S_{t_1} S_{t_2}]$

$= S_0^2 e^{u(t_1+t_2)} E[e^{(-B_{t_1} + \sigma(B_{t_1} + (B_{t_2} - B_{t_1})))}]$

$= S_0^2 e^{u(t_1+t_2)} E[\exp(2\sigma B_{t_1} + \sigma(B_{t_2} - B_{t_1}))]$

$= S_0^2 e^{u(t+t_2)} E[\exp(2\sigma B_{t_1})] E[\exp(B_{t_2} - B_{t_1})]$

$= S_0^2 e^{u(t_1+t_2)} \exp(\frac{3}{2}\sigma^2 t_1 + \frac{1}{2}\sigma^2 t_2)$.

Putting all the equations together.

$Cov[S_{t_1}, S_{t_2}] = S_0^2 e^{u(t_1+t_2)} e^{(3/2 \sigma^2 t_1 + \frac{1}{2}\sigma^2 t_2)}$
$- S_0 e^{u t_1 + 1/2 \sigma^2 t_1}$

7.

$$\begin{tikzcd}[column sep=large] & S_0 u \\ S_0 \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\ & S_0 d \end{tikzcd}$$

$E(C_t) = S_0 (pu + (1-p)d]$.

and $Var[C_t] = E[C_t^2] - E(C_t)^2$

$= S_0^2 [pu^2 + (1-p)d^2] - S_0^2 (pu + (1-p)d]^2$

$= S_0^2 [pu^2 + (1-p)d^2 - (pu + (1-p)d)^2]$

$= S_0^2 (p(1-p)u^2 + p(1-p)d^2 - 2p(1-p))$.

$= S_0^2 p(1-p)(u-d)^2$.

$S_0 e^{rt} = S_0 (pu + (1-p)d)$ $\quad - (1)$

And $\sigma^2 S_0^2 t = S_0^2 p(1-p)(u-d)^2$ $\quad - (2)$.

from i we get.

$p = \frac{e^{rt} - d}{u - d}$

Putting P in equation 2.

$$\sigma^2 t = \frac{e^{rt}-d}{u-d}\left(1-\frac{e^{rt}-d}{u-d}\right)(u-d)^2$$

$$= -(e^{rt}-d)(e^{rt}-u) = (u+d)e^{rt}-(1+e^{2rt}).$$

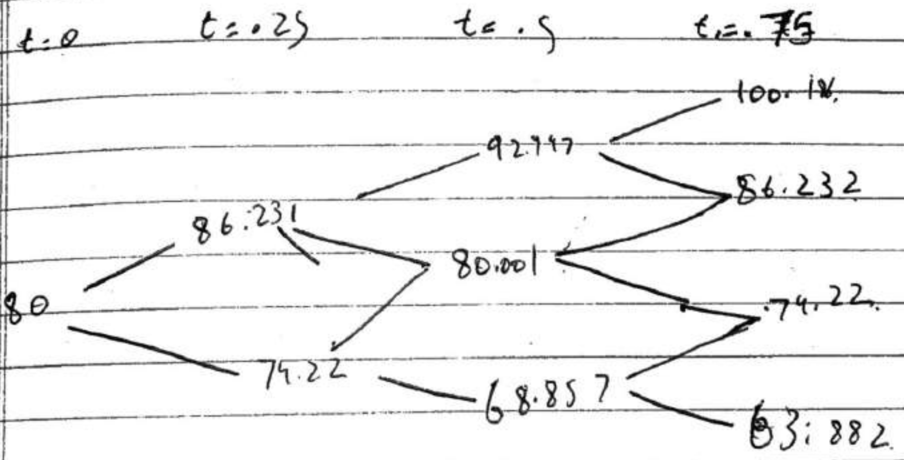
Putting $d = 1/u$ and multiplying by u

$$u^2e^{rt} - u(1+e^{2rt}+\sigma^2 t)+e^{rt} = 0$$

This is a quadratic in u which can be solved

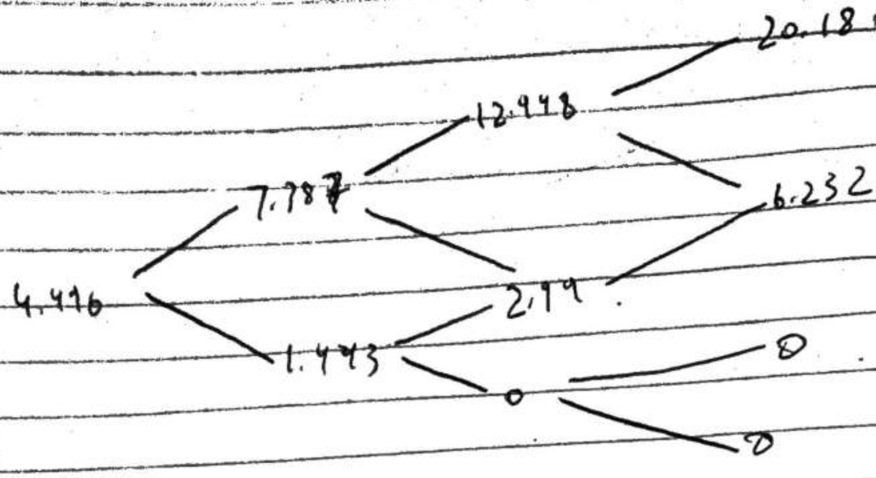
(ii) a) $\sigma = 0.15,\ t = 0.25 \Rightarrow u = e^{(0.15\sqrt{.25})} = e^{0.75} = 1.077884$

$d = 1/u = 0.92774.$



b) $r = 0$; we have $p = \frac{e^{rt}-d}{u-d} = \frac{(1-0.927744)}{1.077884-0.927744} = 0.48126$

Discounting back the final payoff at $t = 0.75$ to $t=0$ along the tree using P and (1-P) we get



Hence value of call option is 4.496

c The lookback call pays the difference between the minimum value and the final value,
Notate paths be U for up and D for down is

	Payoff	Probable
UUU	$(100.186-80)=20.186$	$p^3 = .11147$
UDU	$(86.232-80)=6.232$	$p^2(1-p) = .12015$
UUD	$(86.232-80)=6.232$	$p^2(1-p) = 0.12015$
UDD	$(74.22-74.22)=0$	$p(1-p)^2 = 0.12950$
DUU	$(86.232-74.22)=12.012$	$p^2(1-p) = 0.12015$
DUD	$(74.22-74.22)=0$	$p(1-p)^2 = 0.1290$
DDU	$(74.22-68.857)=5.363$	$p(1-p)^2 = 0.12950$
DDD	$(63.882-63.882)=0$	$p(1-p)^3 = 0.13959$

~~Probabilities~~

$(0.11147 \times 20.186) + (0.12015 \times (6.232+6.232+12.012)$
$+ (0.12950 \times 5.363) = 5.8854.$

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8. (i) Consider a stock whose current price is S_0 and an option whose current price is f. We suppose that the option lasts for time T and that during the life of the option the stock price can either move up from S_0 to a new level S_0u or move down to S_0d where $u > 1$ and $d < 1$.
Let the payoff be f_u if the stock price becomes S_0u and f_d if stock price becomes S_0d.
Let us construct a portfolio which consist of a short position in the option and a long position in shares we calculate the value of Δ that makes the portfolio risk-free.
Now if there is an upward movement in the stock, the value of the portfolio becomes $\Delta S_0u - f_u$ and if there is a downward movement of stock the value of the portfolio becomes $\Delta S_0d - f_d$.
The two portfolios are equal if $\Delta S_0u - f_u = \Delta S_0d - f_d$.
or $\Delta = f_u - f_d / S_0u - S_0d$ so that the portfolio is risk free and hence must earn the risk free rate of interest.
This means the present value of such a portfolio is $(\Delta S_0u - f_u) e^{(-rT)}$,
Where r is the risk free rate of interest.
The cost of the portfolio is $\Delta S_0 - f$.
Since the portfolio grows at a risk free rate it follows that
$$(\Delta S_0u - f_u) e^{(-rT)} = \Delta S_0 - f$$
$$\text{or } f = (\Delta S_0 - f_u) e^{(-rT)}$$
substituting Δ from the earlier equation simplifies to.
$$F = e^{-rT} [p f_u + (1-p) f_d) \text{ where } p = \left(\frac{e^{rt} - d}{u - d}\right).$$

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(ii) The option pricing formula does not involve probabilities of stock going up or down although it is natural to assume that the probabilities of an upward movement in stock increase the value of call option and the value of put option decreases, when the probability of stock price goes down.
This is because we are calculating the value of option not in absolute terms but in terms of the value on the underlying stock already incorporated in the price of the stock, However, it is natural to interpret p as the probability of an up movement in the stock price, The variable $1-p$ is then the probability of a down movement such that above equation can be interpreted as that the value of option Today is the expected future value discounted at the risk-free rate.

(iii) $E(S_T) = pS_0u + (1-p)S_0d$ 0.5

$$p = \frac{(e^{rt}-d)}{(u-d)}$$ we get $E(S_T) = e^{rT}S_0$.

~~i.e stock~~.

(iv) In a risk neutral world individuals do not require to compensation for risk or they are indifferent to risk, Hence expected return on all securities and options is the risk free interest rate. Hence value of an option is its expected payoff in a risk neutral discount at risk free rate.

9)
(i) The forward price is given by $F = S.e^{rt}$
Put call parity state that $C + K.e^{-rt} = P + S$.
Where C and P are the prices of a european call and put option respectively. with strike K and time to expiry t and S is the current stock price.

By put call parity
$$13.334 + 70e^{(-0.25r)} = 0.120 + S.$$
$$8.869 + 75.e^{(-0.25r)} = 0.568 + S.$$
solving for S on r
$$S = 82 \quad r = 7\%$$
$$F = 82.e^{(0.07 \times 0.25)} = 83.45.$$

(ii) $e^{(r_6 \times 0.5)} = e^{(r_3 \times 0.25)} \times e^{(r_6 \times 0.25)}$.
$r_3 = 7\%$.
$$2.569 + 90 \times e^{(-0.5 \times r_6)} = 7.909 + 82$$
Therefore $r_6 = 6\%$ $r_f = 5\%$

(iii) Using Put call Parity
$$6.899 + a \times e^{(-0.07 \times 0.25)} = 1.789 + 82$$
$$2.594 + 85 \times e^{(-0.07 \times 0.25)} = c + 82.$$
$$a = 77.5 \quad b = 5.177 \quad c = 4.119.$$

10) $q = (1-d) / u-d$
(i) $d = 1/u$.
$$q = (1 - 1/u) / (u - 1/u) = 1/u+1$$
for no arbitrary we have $u > 1 > d$.

Then $u > 1 \geq u+1 > 2 \geq q = 1/u+1 < 1/2$ hence proved

(ii) a 12 step recombining binomial tree needs to be created i.e $n = 12$.

Further at time $T = 12$ months, the stock Price will be $S_0 U^K d^{n-K}$ is

${}^nC_K\, q^K (1-q)^{n-K}$ where q, the up step Probability is $1/3$. u

The up-step size is 2

and $d = 1/u = 1/2$

The pay off of derivative is $\sqrt{\frac{S_T}{S_0}}$ at time $T = 12$ months.

Thus $P = \sum_{K=0}^{n} \sqrt{\frac{S_T}{S_0}} \cdot \frac{n!}{K!(n-K)!} q^K (1-q)^{n-K}$.

$$P = \sum_{K=0}^{n} \sqrt{\frac{S_0 U^K d^{n-K}}{S_0}} \cdot \frac{n!}{K!(n-K)!} q^K (1-q)^{n-K}$$

$$= \sum_{K=0}^{n} \sqrt{U^K d^{n-K}} \cdot \frac{n!}{K!(n-K)!} q^K (1-q)^{n-K}$$

$$P = \sum U^{\frac{K}{2}} d^{\frac{n-K}{2}} \cdot \frac{n!}{K!(n-K)!} q^K (1-q)^{n-K} = \sum_{K=0}^{n} 2^{\frac{K}{2}} \left(\frac{1}{2}\right)^{\frac{n-K}{2}} {}^nC_K \left(\frac{1}{3}\right)^K \left(\frac{2}{3}\right)^{n-K}$$

$$P = \sum_{K=0}^{n} 2^{K - \frac{n}{2}} \cdot \frac{n!}{K!(n-K)!} \frac{2^{n-K}}{3^n} = \sum_{K=0}^{n} 2^{\frac{n}{2}} \cdot \frac{n!}{K!(n-K)!} \cdot \frac{1}{3^n}$$

$$= 2^{n/2} \cdot \frac{1}{3^n} \sum_{K=0}^{n} \frac{n!}{K!(n-K)!}$$

$$P = 2^{n/2} \frac{1}{3^n} 2^n = \left(\frac{2\sqrt{2}}{3}\right)^n = \left(\frac{2\sqrt{2}}{3}\right)^{12} = 0.49327$$

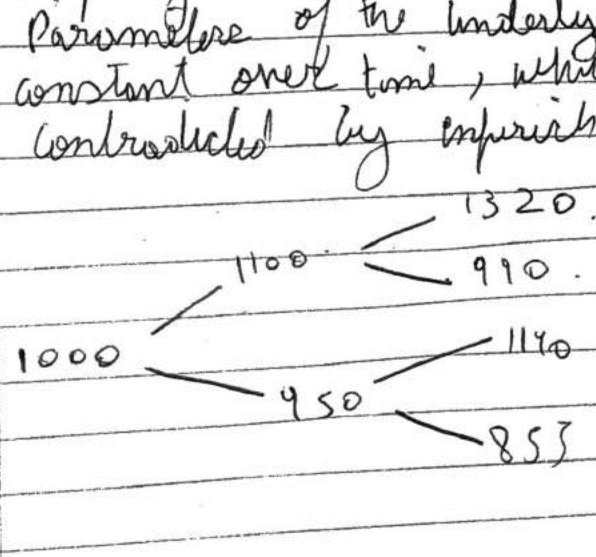
II

(i) A recombining binomial tree or binomial lattice is one in which the sizes of those up step and down steps are assumed to be the same under all states and across all times i.e $u_t(j) = u$ and $d_t(j) = d$ for all times t and states j with $d < e^{(r)} < u$.

It therefore follows that the risk-neutral probability 'q' is also constant at all times and in all stages eg. $q_t(j) = q$

The main advantage of a 'n' period recombining binomial tree is that it has only $n+1$ possible states of time as opposed to 2^n possible states in a similar non-recombining binomial tree. This greatly reduces the amount of computation time required when using a binomial tree model.

The main disadvantage is that the recombining tree implicitly assumes that the volatility and drift parameters of the underlying asset are constant over time, which assumption is contradicted by empirical evidence.



(ii) $q_1 = (e^{0.0175} - 0.95) / (1.10 - 0.95) = 0.0.0676 / 0.15) = 0.4510$

a) $q_2 = e^{(0.025)} - 0.90 / (1.20 - 0.90) = 0.41772.$

Put payoffs at the expiration date at each of the four possible states of expiry are 0, 0, 0 and 95

Working backward, the value of the option $V_1(1)$ following an up step over the first 3 months is

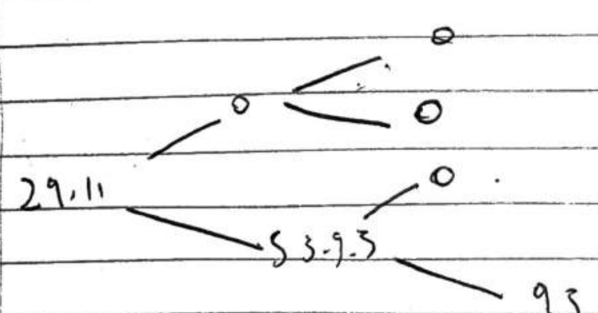
$V_1 e^{(0.025)} = [0.41772 \times 0] + [0.58228 \times 0]$ i.e

$V(1) = 0$

The value of the option $V_1(2)$ following a down step the first 3 months is $V_1(2) e^{(0.025)} = [0.41772 \times 0]$
$+ [0.58228 \times 95]$ i.e.

$V_1(2) = 53.9508$

The current value of the Put option is
$V_0 e^{(0.0175)} = (0.4510 \times 0] +$
$[0.5490 \times 53.9508]$ ie $V_0 = 29.105$



b) While the proposed modification would produce a more accurate valuation there would be a lot more parameter values to specify. Appropriate values of u and d be a required for each branch of the tree and values of 'r' for each month would be required.

The new tree would have $2^6 = 64$ nodes in the expiry column. This would render the calculation probabilities to do normally, and would be a 6 step recombining tree which would have only 7 nodes in the final column.

12. Given $Z(t)$ is Standard Brownian.

a) $dU(t) = 2dZ(t) - 0 = 0dt + 2\,dZ(t)$.
Thus the stochastic process $\{U(t)\}$ has zero drift.

b) $dV(t) = d[Z(t)]^2 - dt \cdot d[Z(t)]^2 = 2Z(t)\,dZ(t) + 2/2\,[dZ(t)]^2 = 2Z(t)dZ(t) + dt$.
by the multiplication rule

Thus $dV(t) = 2Z(t)\,dZ(t)$. The stochastic process $\{V(t)$ has zero drift.

c) $dW(t) = d[t^2 Z(t)] - 2t\,Z(t)\,dt$,
Because $d[t^2 Z(t)] = t^2\,dZ(t) + 2t\,Z(t)\,dt$.
we have $dW(t) = t^2 dZ(t)$
Thus the process $\{W(t)\}$ has zero drift.