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Probability and Statistics Assignment - 2

1. Let X_i be claim sizes ~~of~~ on a home insurance policy.

Let Y_i be claim sizes on a car insurance policy.

$X \sim N(800, (100)^2)$ } Given
$Y \sim N(1200, (300)^2)$

$P(Y_1+Y_2+Y_3 > X_1+X_2+X_3+X_4+800)$ } Given

$P(Y_1+Y_2+Y_3-(X_1+X_2+X_3+X_4) > 800)$

$(Y_1+Y_2+Y_3)-(X_1+X_2+X_3+X_4) \sim N(3\times1200 - 4\times800, 3(300)^2+4$

$(Y_1+Y_2+Y_3)-(X_1+X_2+X_3+X_4) \sim N(400, 310000)$

$P\left(Z > \frac{800-400}{\sqrt{310000}}\right) = P(Z > 0.7184212081)$

$\Rightarrow 1-P(Z < 0.7184212081)$

$= 1 - 0.7637521533$

$\Rightarrow \boxed{0.2362478467}$

$P(Z<0.71) = 0.76115$

$P(Z<0.72) = 0.76424$

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2. N = no of claims

$N \sim$ Neg. Bin. (r, p) $\quad P(N=n) = {}^{n+2}C_n (0.9)^3 (0.1)^n$

$= {}^{(n+3)-1}C_{n-1} (0.9)^3 (0.1)^n$

So, $r=3$, $p=0.9$

$X \sim$ Gamma $(6,2)$ $\quad M_Y(t) = M_N(\log M_X t)$

Total claim amount $= Y$

i) MGF of Y $\quad M_Y(t) = E(e^{X_1 t + X_2 t + X_3 t \dots + X_n(t)})$

$= E(e^{tX_1}).E(e^{tX_2})\dots E(e^{tX_n})$

$= \left(1 - \frac{t}{2}\right)^{-6n}$

So, $E(E(e^{Yt}/N=n)) = E\left(\left(1-\frac{t}{2}\right)^{-6n}\right)$

$= E\left(e^{N \log\left(1-\frac{t}{2}\right)^{-6}}\right)$

$= M_N\left(\log\left(1-\frac{t}{2}\right)^{-6}\right)$

$M_N(t) = \left(\frac{pe^t}{1-qe^t}\right)^k = \left(\frac{pe^{\log(1-t/2)^{-6}}}{1-qe^{\log(1-t/2)^{-6}}}\right)^k$

$\Rightarrow \left[\frac{(0.9)\left(1-\frac{t}{2}\right)^{-6}}{1-(0.1)\left(1-\frac{t}{2}\right)^{-6}}\right]^3$

$V(Y) = E(N)V(X) + (E(X))^2 . V(N)$

$= \left[\frac{3}{0.9} \times \frac{6}{4}\right] + \left[\left(\frac{6}{2}\right)^2 \times \frac{3 \times 0.1}{(0.9)^2}\right]$

$= \frac{18}{3.6} + \left(\frac{9 \times 0.3}{0.81}\right)$

$= 5 + 3.33333$

$= 8.33333333$

S.D. $= \sqrt{V(Y)}$

S.D. $\Rightarrow$ 2.886751346

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Q3. $f(x) = \lambda e^{-\lambda(x-\alpha)}$, $x \geq \alpha$, $\lambda, \alpha > 0$

i) $MGF = E(e^{tx}) = \int_{\alpha}^{\infty} e^{tx} \lambda e^{-\lambda(x-\alpha)} dx$

$M_x(t) = \lambda \int_{\alpha}^{\infty} e^{tx} e^{-\lambda x + \lambda \alpha} dx = e^{\lambda a} \int_{\alpha}^{\infty} e^{-(\lambda - t)x} dx$

$= \frac{e^{\alpha\lambda}\lambda}{t-\lambda} \left[e^{-(\lambda-t)x} \right]_{\alpha}^{\infty} = \frac{e^{\lambda\alpha}\lambda}{t-\lambda} \left[-e^{-(\lambda-t)\alpha} \right.$

$= \frac{e^{\lambda a}\lambda}{\lambda - t} e^{-\lambda a} e^{t\alpha} \Rightarrow \boxed{e^{t\alpha} \frac{\lambda}{\lambda - t}}$

ii) $M_x(t) = \frac{\lambda e^{t\alpha}}{\lambda - t} = \lambda e^{t\alpha} (\lambda - t)^{-1}$

$M_x'(t) = \lambda [\alpha e^{t\alpha} (\lambda - t)^{-2} + \alpha^2 (\lambda - t)^{-1} e^{t\alpha} + 2e^{t\alpha} (\lambda - t)^{-3}$
$+ \alpha e^{t\alpha} (\lambda - t)^{-2}]$

Put $t = 0$

$M_x'(0) = \lambda [\alpha(\lambda)^{-2} + \alpha^2 (\lambda)^{-1} + 2 (\lambda)^{-3} + \alpha (\lambda)^{-2}]$

$= \lambda \left[\frac{2\alpha}{\lambda^2} + \frac{\alpha^2}{\lambda} + \frac{2}{\lambda^3} \right]$

$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2}$

$V(X) = E(X^2) - [E(X)]^2 = \left[\frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} \right] - \left[\alpha + \frac{1}{\lambda} \right]^2$

$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left(\alpha^2 + \frac{1}{\lambda^2} + \frac{2\alpha}{\lambda} \right)$

$= \cancel{\frac{2\alpha}{\lambda}} + \cancel{\alpha^2} + \frac{2}{\lambda^2} - \cancel{\alpha^2} - \frac{1}{\lambda^2} - \cancel{\frac{2\alpha}{\lambda}}$

$\boxed{V(X) = \frac{1}{\lambda^2}}$

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4. $G_V(t) = \left(\frac{p}{1-qt}\right)^k \quad , p+q=1$

$G_V(t) = p^k(1-qt)^{-k} = \sum t^v \times P(V=v)$

$P(V=4) = \frac{\Gamma k+4}{\Gamma 5 \, \Gamma k} p^k q^k$

$= \frac{\Gamma k+4}{4! \, \Gamma k} p^k q^k$

$\Rightarrow \frac{(k+3)!}{(k-1)! \, 4!} p^k (1-p)^k$

5. $f(x,y) = ax(x+y)$

$P(4 > \frac{1}{3}X + 10)$

$$a\int_{10}^{20}\int_{0}^{5}(x^2+xy)\,dx\,dy = 1$$

$$\int_{10}^{20} \frac{x^3}{3} + \frac{x^2}{2}y \Big|_0^5 \, dy = \frac{1}{a}$$

$$\int_{10}^{20}\left(\frac{125}{3} + \frac{25}{2}y\right)dy = \frac{1}{a}$$

$$\frac{125}{3}y + \frac{25}{4}y^2 \Big|_{10}^{20} = \frac{1}{a}$$

$$a\left[\frac{2500}{3} + 2500 - \frac{1250}{3} - 625\right] = 1$$

$$a\left(\frac{1250 + 1875}{3}\right) = 1$$

$$a\left(\frac{6875}{3}\right) = 1$$

$$a \Rightarrow \frac{3}{6875} \Rightarrow 0.00043636$$

$E(Y/X)$

$$f_{Y/X} = \frac{f(x,y)}{f(x)}$$

$$f(x) = \frac{3}{6875}\int_{10}^{20}(x^2+xy)\,dy = \frac{3}{6875}\left[x^2y + \frac{xy^2}{2}\right]_{10}^{20}$$

$$= \frac{3}{6875}\left[20x^2 + x(200) - 10x^2 - 50x\right] = \frac{3}{6875}\left[10x^2 + 150x\right]$$

$$= \frac{30x}{6875}\left[x+15\right]$$

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$$f_{y/x} = \frac{3}{6875} \cdot \frac{x(x+y)}{30x} \cdot \frac{6875}{(x+15)} \Rightarrow \frac{\cancel{3}}{\cancel{6875}} \cdot \frac{\cancel{x}(x+y)}{10\cancel{x}} \cdot \frac{\cancel{6875}}{x+15}$$

So, $E(y/x) = \int_{10}^{20} \frac{y(x+y)}{10(x+15)}\,dy = \frac{1}{10(x+15)} \int_{10}^{20} (xy+y^2)\,dy$

$$\frac{1}{10(x+15)}\left[\frac{xy^2}{2}+\frac{y^3}{3}\right]_{10}^{20} = \frac{1}{10(x+15)}\left[200x+\frac{8000}{3}-50x-\frac{1000}{3}\right]$$

$$\frac{1}{10(x+15)}\left[150x+\frac{7000}{3}\right]$$

$$\Rightarrow \frac{1}{x+15}\left[\frac{15x+700}{3}\right]$$

So, $\frac{3}{6875}\int_{\frac{1}{3}x+10}^{20}\left(\frac{125}{3}+\frac{25y}{2}\right)dy = \frac{3}{6875}\left[\frac{125y}{3}+\frac{25}{4}y^2\right]_{\frac{1}{3}x+10}^{20}$

$$\frac{3}{6875}\left[\frac{125}{3}(20)+\frac{25}{4}(400)-\frac{125}{3}\left(\frac{1}{3}x+10\right)-\frac{25}{4}\left(\frac{1}{3}x+10\right)^2\right]$$

$$\frac{3}{6875}\left[\frac{2500}{3}+2500-\frac{125x}{9}-\frac{1250}{3}-\frac{25}{4}\left[\frac{1}{9}x^2+100+\frac{20x}{3}\right]\right]$$

$$\frac{3}{6875}\left[\frac{2500}{3}+2500-\frac{125x}{9}-\frac{1250}{3}-\frac{25x^2}{36}-\frac{2500}{4}-\frac{500x}{12}\right]$$

$$\frac{3}{6875}\left[\frac{6875}{3}-\frac{125x}{9}-\frac{25x^2}{36}-\frac{500x}{12}\right]$$

$$\frac{3}{6875}\left[\frac{6875}{3}-\frac{500x-25x^2-1500x}{36}\right]$$

$$\Rightarrow 1-\frac{500x-25x^2-1500x}{82500}$$

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6. $X \sim Poi(\lambda)$ $\quad Y \sim Poi(\mu)$

$M_X(t) = e^{\lambda(e^t-1)}$ $\quad M_Y(t) = e^{\mu(e^t-1)}$

So, $Z = X - Y$

$M_Z(t) = E(e^{tZ}) = E(e^{t(X-Y)}) = E(e^{tX} \cdot e^{-tY})$

$= E(e^{tX}) \cdot E(e^{-tY}) = \frac{E(e^{tX})}{E(e^{tY})} = \frac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}}$

$= e^{(\lambda-\mu)(e^t-1)}$

By uniqueness of Z's MGF $= e^{(\lambda-\mu)(e^t-1)} \sim Poi(\lambda-\mu)$

Now, Let $Z = X+Y$

$E(e^{tZ}) = e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)}$

$= e^{(\lambda+\mu)(e^t-1)}$

So, By uniqueness $Z \sim Poi(\lambda+\mu)$.

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7.

			X		
		1	2	3	4
y	1	0.2	0	0.05	0.15
	2	0	0.3	0.1	0.2

i) $V(X/Y=2) = E(X^2/Y=2) - [E(X/Y=2)]^2$

Now, $E(X^2/Y=2) = \sum_x x^2 \frac{P(X=x, Y=2)}{P(Y=2)}$

$= \frac{(1)^2 P(X=1, y=2)}{P(y=2)} + \frac{(2)^2 P(X=2, y=2)}{P(y=2)} + \frac{(3)^2 P(X=3, y=2)}{P(y=2)} + \frac{(4)^2 P(X=4, y=2)}{P(y=2)}$

$= \frac{(1)(0)}{0.6} + \frac{4 \times 0.3}{0.6} + \frac{9 \times 0.1}{0.6} + \frac{16 \times 0.2}{0.6}$

$\Rightarrow 8.8333333$

Now, $E(X/Y=2) = \sum_x x \frac{P(X=x, Y=y)}{P(Y=2)}$

$= \frac{(1)(0)}{0.6} + \frac{2 \times 0.3}{0.6} + \frac{3 \times 0.1}{0.6} + \frac{4 \times 0.2}{0.6}$

$= \frac{0.6 + 0.3 + 0.8}{0.6} = 2.833333$

$V(X/Y=2) = (8.83333) - (2.83333)^2$

$\underline{V(X/Y=2) = 0.805555}$

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ii) $f(u,v) = \frac{48}{67}(2uv - u^2)$, $0 < u < 1 ;\ \frac{u}{2} < v < 2$

$E(U/V = v) = ?$

$\frac{f(u,v)}{f(v)} = f(u/v) = v$

$f(v) = \frac{48}{67}\int_0^1 (2uv - u^2)\,du = \frac{48}{67}\left[u^2 v - \frac{u^3}{3}\right]_0^1$

$\frac{48}{67}\left[v - \frac{1}{3}\right] = \frac{16}{67}[3v-1]$

$f(v) = \frac{16}{67}[3v-1]$

$\frac{f(u,v)}{f(v)} = \frac{\cancel{48}\,3(2uv^2 - u^2)}{\cancel{67}\ \cancel{16}(3v-1)} \times \cancel{67}$

$= \frac{3}{3v-1}(2uv^2 - u^2)$

$E(U/V = v) = \frac{3}{3v-1}\int_0^1 u(2uv^2 - u^2)\,du$

$\frac{3}{3v-1}\left[\frac{2u^3v^2}{3} - \frac{v^4}{4}\right]_0^1 = \frac{3}{3v-1}\left[\frac{2v^2}{3} - \frac{1}{4}\right]$

$\frac{\cancel{3}}{3v-1}\left[\frac{8v^2-3}{\cancel{12}\,4}\right] \Rightarrow \frac{8v^2-3}{4(3v-1)}$

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8. $X \sim Gamma(3,2)$

$E(X) = \frac{3}{2}$ $\qquad V(X) = \frac{3}{2^2} = \frac{3}{4}$

$E(Y/X=x) = 3x+1$ $\qquad Var(Y/X=x) = 2x^2+5$

$E(Y) = E(E(Y/X=x)) = E(3x+1)$

$= 3E(x)+1$

$= 3\left(\frac{3}{2}\right)+1 = \frac{9}{2}+1 \Rightarrow \frac{11}{2}$

$V(y) = E(V(X/y)) + V(E(X/Y))$

$= E(2x^2+5) + V(3x+1)$

$2E(X^2)+5+9V(x)$

$V(X) = E(X^2) - (E(X))^2$

$E(X^2) = \frac{3}{4} + \frac{9}{4} = \frac{12}{4} = 3$ $\qquad \boxed{E(X^2)=3}$

$V(Y) = 2(3)+5+9\left(\frac{3}{4}\right) = \frac{44+27}{4} = \frac{71}{4}$ $\qquad \boxed{V(Y) = \frac{71}{4}}$

9. $E(X)=3$ $V(X)=4$
$E(Y)=4$ $V(Y)=1$

Correlation Coefficient $= -0.3\sqrt{4} = cov(X,Y)$
$= -0.6$

$Z = X+Y$

i) $cov(X, X+Y) = cov(X,X) + cov(X,Y)$
$= V(X) + cov(X,Y)$
$= 4 - 0.6$
$\Rightarrow 3.4$

ii) $Var(Z) = cov(X+Y, X+Y)$
$= cov(X,X) + 2\,cov(X,Y) + cov(Y,Y)$
$= V(X) + 2\,cov(X,Y) + V(Y)$
$= 4 + 2(-0.6) + 1$
$= 5 - 1.2$
$\Rightarrow 3.8$

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10. $f(x,y) = Kx^{-\alpha} e^{-y/\beta}$

$1 < x < \infty \;, \; 1 < y < \infty$

$\alpha > 1 \;, \; \beta > 0$

$K \int_1^\infty \int_1^\infty x^{-\alpha} e^{-y/\beta} \, dx \, dy = 1$

$K \int e^{-y/\beta} \left[\frac{x^{-(\alpha-1)}}{-\alpha+1} \right]_1^\infty dy = 1$

$\frac{K}{1-\alpha} \int e^{-y/\beta} (-1) \, dy = 1$

$-\frac{K}{1-\alpha} \int e^{-y/\beta} \, dy = 1$

$\frac{K}{\alpha-1} \left[\frac{e^{-y/\beta}}{(-1/\beta)} \right]_1^\infty$

$\frac{K}{\alpha-1} (\beta) (e^{-1/\beta}) = 1$

$$\boxed{K = \frac{\alpha-1}{(\beta)(e^{-1/\beta})}}$$