Probablity & Stats - 1

Assignment - 2

① X = claim size of house insurance.
Y = claim size of car insurance

$X \sim$ Normal $(800, 100^2)$
$Y \sim$ Normal $(1200, 300^2)$

$P(Y_1 + Y_2 + Y_3 > X_1 + X_2 + X_3 + X_4 + 800)$
$= P((Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) > 800)$.

$\therefore (Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) \sim$
$N(3(1200) - 4(800),$
$(3 \times 300^2 + 4 \times 100^2)$

$\Rightarrow (Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) \sim N(400, 310000)$

$\therefore P((Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) > 800)$
$= P\left(Z > \frac{800 - 400}{\sqrt{310000}}\right)$ $= P(Z > 0.718)$
$= 1 - P(Z < 0.718)$
$= 1 - 0.76424$
$= 0.23576$

② N = number of claims.
$N \sim$ Neg Bin $(3, 0.9)$.
X = Claim Amount
$X \sim$ Gamma $(6, 2)$
Y = Total claim amount.
~~$M_Y(t) = M_N(\log(\ldots))$~~

9) $M_x(t) = \left(1 - \frac{t}{2}\right)^{-6}$

$M_y(t) = M_N(\log M_x(t))$

$= M_N\left(\log\left(1-\frac{t}{2}\right)^{-6}\right)$

$M_N(t) = \left(\frac{pe^t}{1-qe^t}\right)^r$

$\therefore M_y(t) = \left(\frac{pe^{\log\left(1-\frac{t}{2}\right)^{-6}}}{1-qe^{\log\left(1-\frac{t}{2}\right)^{-6}}}\right)^3$

$= \left(\frac{p\left(1-\frac{t}{2}\right)^{-6}}{1-q\left(1-\frac{t}{2}\right)^{-6}}\right)^3$

$M_y(t) = \frac{0.9\left(1-\frac{t}{2}\right)^{-6}}{1-0.1\left(1-\frac{t}{2}\right)^{-6}}$

$E(N) = \frac{3}{0.9} = 3.3333$

$V(N) = \frac{0.1\times 3}{(0.9)^2} = \frac{0.3}{0.81} = 0.37037$

$E(X) = \frac{6}{2} = 3$, $V(X) = \frac{6}{4} = \frac{3}{2} = 1.5$

$V(Y) = E(N)\cdot V(X) + (E(X))^2 \cdot V(N)$

$= 3.333\times 1.5 + (3)^2(0.37037)$

$= 5 + 3.333 = 8.333$

$$SD(4) = \sqrt{8.333}$$

$$= 2.88675$$

Q3 $f_x(x) = \lambda e^{-\lambda(x-\alpha)}$

a) $M_x(t) = E(e^{tx})$

$$= \int_{\alpha}^{\infty} e^{tx} \lambda e^{-\lambda(x-\alpha)} \, dx$$

$$= \lambda \int_{\alpha}^{\infty} e^{tx} \cdot e^{-\lambda(x-\alpha)} \, dx$$

$$= \lambda \int_{\alpha}^{\infty} e^{tx - \lambda x + \lambda \alpha} \, dx$$

$$= \lambda \int_{\alpha}^{\infty} e^{tx - \lambda x} \cdot e^{\lambda \alpha} \, dx$$

$$= \lambda e^{\lambda \alpha} \int_{\alpha}^{\infty} e^{x(t-\lambda)} \, dx$$

$$= \lambda e^{\lambda \alpha} \left(\frac{e^{x(t-\lambda)}}{t-\lambda} \right)_{\alpha}^{\infty}$$

$$= \frac{\lambda e^{\lambda \alpha}}{t-\lambda} \left(e^{x(t-\lambda)} \right)_{\alpha}^{\infty}$$

$$= \frac{\lambda e^{\lambda \alpha}}{t-\lambda} \left[0 - e^{\alpha t - \alpha \lambda} \right]$$

$$= \frac{\lambda e^{\lambda \alpha}}{t-\lambda} \left[- e^{-\lambda \alpha} \cdot e^{\alpha t} \right]$$

$$= \frac{-\lambda e^{t\alpha}}{t-\lambda} = \frac{\lambda e^{t\alpha}}{\lambda - t} = M_x(t)$$

(ii) ~~E(x)~~ $M'_x(t) = \lambda(e^{t\alpha})(\lambda-t)^{-2}(-1)(-1) + \lambda(\lambda-t)^{-1}(e^{t\alpha})(\alpha)$

$$= \frac{\lambda e^{t\alpha}}{(\lambda-t)^2} + \frac{\lambda e^{t\alpha}\alpha}{\lambda-t}$$

$$E(x) = M'_x(0) = \frac{\lambda(1)}{\lambda^2} + \frac{\lambda\alpha}{\lambda}$$

$$= \frac{\lambda+\lambda^2\alpha}{\lambda^2}$$

$$= \frac{1+\lambda\alpha}{\lambda}$$

$$M''_x(t) = \lambda e^{t\alpha}(-2)(\lambda-t)^{-3}(-1) + \lambda(\lambda-t)^{-2}(e^{t\alpha})(\alpha) + \alpha\lambda e^{t\alpha}(\lambda-t)^{-2}(-1)(-1) + \lambda\alpha(\lambda-1)^{-1}(e^{t\alpha})(\alpha)$$

$$= \frac{2\lambda e^{t\alpha}}{(\lambda-t)^3} + \frac{\lambda\alpha e^{t\alpha}}{(\lambda-t)^2} + \frac{\lambda\alpha e^{t\alpha}}{(\lambda-t)^2} + \frac{\alpha^2\lambda e^{t\alpha}}{\lambda-t}$$

$$= \frac{2\lambda e^{t\alpha}}{(\lambda-t)^3} + \frac{2\lambda\alpha e^{t\alpha}}{(\lambda-t)^2} + \frac{\alpha^2\lambda e^{t\alpha}}{\lambda-t}$$

$$E(x^2) = M''_x(0) = \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2$$

$$V(x) = E(x^2) - (E(x))^2$$

$$= \alpha^2 + \frac{2\alpha}{\lambda} + \frac{2}{\lambda^2} - \left(1 + \frac{\lambda^2\alpha^2 + 2\alpha\lambda}{\lambda^2}\right)$$

$$= \cancel{\alpha^2} + \cancel{\frac{2\alpha}{\lambda}} + \frac{2}{\lambda^2} - \frac{1}{\lambda^2} - \cancel{\alpha^2} - \cancel{\frac{2\alpha}{\lambda}}$$

$$V(x) = \frac{1}{\lambda^2}$$

Q4 $G_x(t) = \left(\frac{p}{1-qt}\right)^k$, $p + q = 1$

$\therefore$ Distribution is negative binomial

$$P(X = 4) = \frac{\sqrt{k+4}}{\sqrt{5}\,\sqrt{k}} \cdot p^4 \cdot q^4$$

Q5 $f(x, y) = ax^2 + axy$ $\quad 0 < x < 5$

$10 < y < 20$

$$\int_{10}^{20}\int_{0}^{5} ax^2 + axy \, dx \, dy = 1$$

$$\int_{10}^{20} \left(\frac{ax^3}{3}\right)_0^5 + \left(ay\frac{x^2}{2}\right)_0^5 dy = 1$$

$$\int_{10}^{20} \frac{125a}{3} + \frac{25}{2}ay \, dy = 1$$

$$\frac{125a}{3}(y)_{10}^{20} + \frac{25}{2}a\left(\frac{y^2}{2}\right)_{10}^{20} = 1$$

$$\frac{1250a}{3} + 1875a = 1$$

$$\frac{1250a + 5625a}{3} = 1$$

$$\frac{6875a}{3} = 1$$

$$a = \frac{3}{6875}$$

$$f_x(x) = \int_{10}^{20} (x^2 + xy)a \, dy$$

$$= \frac{3}{6875} \int_{10}^{20} (x^2 + xy)$$

$$= \frac{3}{6875} \left[x^2 (y)_{10}^{20} + \frac{x (y^2)_{10}^{20}}{2} \right]$$

$$= \frac{3}{6875} [10x^2 + 150x]$$

$$E(Y|X) = y f(Y|X) = y \int_{10}^{20} \frac{f(x,y)}{f_x(x)} dy$$

$$= y \int_{10}^{20} \frac{k(x^2 + xy)}{k(10x^2 + 150x)} dy$$

$$= y \int_{10}^{20} \frac{x(x+y)}{10x(x+15)}$$

$$= \frac{\cancel{y}}{10} \int_{10}^{20} \frac{xy + y^2}{x+15}$$

$$= \frac{\cancel{y} \cdot 1}{10(x+15)} \left(\frac{y^3}{2}\right)_{10}^{20} \left(\frac{xy^2}{2}\right)_{10}^{20}$$

$$= \frac{1}{\cancel{10}(x+15)} \cdot \frac{7000}{\cancel{3}} \div \cancel{150}\,50\,x$$

$$= \frac{35000}{x+15}$$

Q8 $P(Y > \frac{1}{3}x + 10)$

$$f_Y(y) = \int_0^5 \frac{3}{6875}(x^2 + xy)\,dx$$

$$\frac{3}{6875} \int_0^5 x^2 + xy\,dx$$

$$\frac{3}{6875} \left(\frac{x^3}{3}\right)_0^5 + \left(\frac{x^2}{2}\right)_0^5 y$$

$$= \frac{3}{6875}\left[\left(\frac{125}{3}\right) + \frac{25y}{2}\right]$$

$$= \frac{\cancel{3}}{6875}\left[\frac{250 + 75y}{\cancel{6}\,2}\right] = \frac{250 + 75y}{13750}$$

$$P(Y > \tfrac{1}{3}x + 10) = \int_{\frac{1}{3}x+10}^{20} \frac{250 + 75y}{13750}\,dy$$

$$\frac{1}{13750}\left[(250y) + \frac{75y^2}{2}\right]_{\frac{1}{3}x+10}^{20}$$

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$$= \frac{1}{13750}\left[250\left(20 - \frac{x}{3} - 10\right) + 75\left(200 - \frac{x^2}{9} - 100 - \frac{10x}{3}\right)\right]$$

$$= \frac{1}{13750}\left[5000 - \frac{250x}{3} - 2500 + 15000 - \frac{25x^2}{3} - 7500 - \frac{1500x}{3}\right]$$

$$= \frac{1}{13750}\left[10000 - \frac{1750x}{3} - \frac{25x^2}{3}\right]$$

Q8) $X \sim Poi(\lambda)$

$Y \sim Poi(\mu)$

$M_x(t) = e^{\lambda(e^t - 1)}$

$M_y(t) = e^{\mu(e^t - 1)}$

$$M_{x-y}(t) = E(e^{t(x-y)})$$

$$= E(e^{xt - yt}) = \frac{E(e^{xt})}{E(e^{yt})}$$

$$= \frac{e^{\lambda(e^t - 1)}}{e^{\mu(e^t - 1)}} = e^{(\lambda - \mu)(e^t - 1)}$$

$\therefore X - Y \sim Poi(\lambda - \mu)$ $\Rightarrow$ Hence Proved.

$$M_{x+y}(t) = \cancel{E(e^{tx})}\ E(e^{t(x+y)})$$

$$= E(e^{xt + yt}) = E(e^{xt}) \cdot E(e^{yt})$$

$$= e^{\lambda(e^t - 1)} \cdot e^{\mu(e^t - 1)}$$

$$= e^{(\lambda + \mu)(e^t - 1)}$$

$\therefore X + Y \sim Poi(\lambda + \mu)$

Hence proved

Q7 $\operatorname{Var}(X|Y=2) = E(X^2|Y=2) - E(X|Y=2)^2$

$$= \frac{\sum_x x^2 P(X=x, Y=2)}{P(Y=2)} - \left(\frac{\sum_x x P(X=x, Y=2)}{P(Y=2)}\right)^2$$

$$= \frac{1^2(0) + 2^2(0.3) + 3^2(0.1) + 4^2(0.2)}{0.6}$$

$$- \left[\frac{1(0) + 2(0.3) + 3(0.1) + 4(0.2)}{0.6}\right]^2$$

$$= \frac{53}{6} - \frac{289}{36} = \frac{29}{36}$$

(8) $X \sim \text{Gamma}(3, 2)$

$E(Y|X=x) = 3x + 1$

$V(Y|X=x) = 2x^2 + 5$

$E(E(Y|X=x)) = E(Y)$

$E(Y) = E(3X+1)$

$= 3E(X) + 1$

$= 3\left(\frac{3}{2}\right) + 1$

$= \frac{9}{2} + 1$

$= \frac{11}{2}$

$\overset{Ve}{\cancel{Var}}(V(Y|X=x)) = V(Y)$

$V(Y) = E(V(Y|X=x)) + V(E(Y|X=x))$

$= E(2x^2+5) + V(3x+1)$

$= 2E(X^2) + 5 + 9V(X)$

$= 2(3) + 5 + 9\left(\frac{3}{4}\right)$

$= 6 + 5 + \frac{27}{4}$

$= 11 + 6.75$

$= 17.75$

(9) $Z = X + Y$

$E(X) = 3$

$E(Y) = 4$

$\sqrt{V(X)} = 2$

$V(X) = 4$

$\sqrt{V(Y)} = 1$

$V(Y) = 1$

i) $Cov(X, Z) = Cov(X, X + Y)$

$= Cov(X, X) + Cov(X, Y)$

$= V(X) + (-0.6)$

$= 4 - 0.6 = 3.4$

ii) $Var(Z) = Var(X + Y)$

$= V(X) + V(Y) + 2\,Cov(X, Y)$

$= 4 + 1 + 2(-0.6)$

$= 5 - 1.2$

$= 3.8$

10) $f(x,y) = kx^{-\alpha} e^{-y/\beta}$, $1 < x < \infty$

$1 < y < \infty$

$\alpha > 1, \beta > 0$

$$k \int_1^\infty x^{-\alpha}\, dx \int_1^\infty e^{-y/\beta}\, dy = 1$$

$$k \left[\frac{x^{-\alpha+1}}{-\alpha+1} \right]_1^\infty \left[\frac{e^{-y/\beta}}{-1/\beta} \right]_1^\infty = 1$$

$$k \left[0 - \frac{1}{-\alpha+1} \right] \left[0 + \beta (e^{-1/\beta}) \right] = 1$$

$$k \left[\frac{1}{-\alpha+1} \right] \left[-\beta e^{-1/\beta} \right] = 1$$

$$k = \cancel{\alpha} \frac{-1+\alpha}{\beta e^{-1/\beta}}$$

$$= \frac{\alpha - 1}{\beta e^{-1/\beta}}$$