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Probablity & Statistics

1) Let x_i be the claim size of home insurance
Let y_i be the claim size of car insurance

Let $X \sim N(800, 100^2)$
$Y \sim N(1200, 300^2)$

$$P(Y_1 + Y_2 + Y_3 > X_1 + X_2 + X_3 + X_4 + 800)$$
$$= P(Y_1 + Y_2 + Y_3 - (X_1 + X_2 + X_3 + X_4) > 800)$$

$$Y_1 + Y_2 + Y_3 - (X_1 + X_2 + X_3 + X_4) \sim N[(3 \times 1200 - 4 \times 800), (300^2(3) + 4$$
$$\sim N(400, 310000)$$

$$P\left(Z > \frac{800 - 400}{\sqrt{310000}}\right)$$

$$P(Z > 0.71842) \longrightarrow 1 - P(Z < 0.71842)$$
$$= 1 - P(Z < 0.72)$$
$$= 1 - 0.76424$$
$$= 0.23576$$

2) $N \sim$ Negative Binomial (k, p)

$$P(N = n) \longrightarrow {}^{\cancel{n+2}\ n+2}C_n (0.9)^3 (0.1)^n$$
$$= {}^{(n+3)-1}C_{n-1} (0.9)^3 (0.1)^n$$

So, $k = n$, $p = 0.1$
$k = 3$, $p = 0.9$ $\qquad M_y t = M_n(\log m_x t)$

$X \sim \text{Gamma}(6, 2)$

$y \rightarrow$ Total claim amount

$E(e^{yt}/N=n) = E(e^{x_1t + x_2t + \cdots\cdots x_nt})$

$= E(e^{tx_1}) \cdot E(e^{tx_2}) \cdots\cdots E(e^{tx_3})$

$= \left(1 - \frac{t}{2}\right)^{-6n}$

So, $E(E(e^{yt}/N=n)) = E\left[\left(1-\frac{t}{2}\right)^{-6n}\right]$

$= E\left(e^{N\log\left(1-\frac{t}{2}\right)^{-6}}\right)$

$= M_N\left[\log\left(1-\frac{t}{2}\right)^{-6}\right]$

$M_N(t) = \frac{pe^t}{1-qe^t} = \left[\frac{pe^{\log\left(1-\frac{t}{2}\right)^{-6}}}{1-qe^{\log\left(1-\frac{t}{2}\right)^{-6}}}\right]^k$

$\because k = 3,\ p = 0.9$

$= \left[\frac{0.9\left(1-\frac{t}{2}\right)^{-6}}{1-0.1\left(1-\frac{t}{2}\right)^{-6}}\right]^3$

ii) $V(Y) = E(N)V(X) + \left[(E(X))^2 \cdot V(N)\right]$

$= \left[\frac{3}{0.9} \times \frac{6}{4}\right] + \left[\left(\frac{6}{2}\right)^2 \times \frac{3 \times 0.1}{(0.9)^2}\right]$

$= 1/0.2 + \left[6 \times 0.3/0.81\right]$

$= 7.\overline{22}$

$SD = \sqrt{\text{var}(y)} = \sqrt{7.\overline{22}} = 2.687419$

3) $y(x) = \lambda e^{-\lambda(x-\alpha)}$

$$M_X F = E(e^{tx}) = \int_{\alpha}^{\infty} e^{tx} \lambda e^{-\lambda(x-\alpha)} dx$$

$$M_X(t) = \lambda \int_{\alpha}^{\infty} e^{tx} e^{-\lambda x} e^{\lambda \alpha} dx$$

$$= e^{\lambda\alpha} \lambda \int_{\alpha}^{\infty} e^{-(\lambda - t)x} dx$$

$$= \frac{e^{\lambda\alpha} \lambda}{t - \lambda} \cdot \left[e^{-(\lambda-t)x} \right]_{\alpha}^{\infty}$$

$$= e^{t\alpha} \left(\frac{\lambda}{\lambda - t} \right)$$

ii) $M_X(t) = \frac{\lambda e^{t\alpha}}{\lambda - t} = \lambda e^{t\alpha} (\lambda - t)^{-1}$

$$M_X'(t) = \lambda \left[e^{t\alpha} (\lambda - t)^{-2} (1) + (\lambda - t)^{-1} e^{t\alpha} (\alpha) \right]$$

$$= \lambda e^{t\alpha} (\lambda - t)^{-2} (\alpha(\lambda - t) + 1)$$

$t = 0$ $= \lambda (\lambda)^{-2} (\alpha(\lambda + 1))$

$$= \frac{\lambda\alpha + 1}{\lambda}$$

$$M''(t) = \lambda \left[\alpha e^{t\alpha} (\lambda - t)^{-2} + \alpha^2 (\lambda - t)^{-1} e^{t\alpha} + e^{t\alpha} (2) (\lambda - t)^{-3} + (\lambda - t)^{-2} (e^{t\alpha}) \right]$$

Put $t = 0$

$$= \lambda \left[\alpha(\lambda)^{-2} + \alpha^2 (\lambda)^{-1} + 2(\lambda)^{-3} + \alpha(\lambda)^{-2} \right]$$

$$= \lambda \left[\frac{2\alpha}{\lambda^2} + \frac{\alpha^2}{\lambda} + \frac{2}{\lambda^3} \right]$$

$$E(X^2) = \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left[\alpha + \frac{1}{\lambda}\right]^2$$

$$= \frac{1}{\lambda^2}$$

4) $G_V(t) = \left(\frac{p}{(1-qt)}\right)^k \quad ; \; p + q = 1$

$$G_V(t) = \sum t^v \cdot P(V = v)$$

$$P(V = 4) = \frac{\sqrt{k+4}}{\sqrt{5}\,\sqrt{k}}\, p^k q^k \qquad \because G_V(t) = p^k (1-qt)^k$$

$$= \frac{\overline{|k+2}}{4!\,\overline{|k}}\, p^k q^k$$

$$= \frac{(k+3)!}{(k-1)!\,4!}\, p^k (1-qp)^k$$

5) $a = ?$

$$a \int_{10}^{20}\int_{0}^{5} (x^2 + xy)\, dx\, dy = 1$$

$$\therefore \int_{10}^{20} \left(\frac{x^3}{3} + \frac{x^2}{2} y\right)_0^5 dy = \frac{1}{a}$$

$$\therefore \left(\frac{125}{3} y + \frac{25}{2} \frac{y^2}{(2)}\right)_{10}^{20} = \frac{1}{a}$$

$\therefore a = 3$ 0.00143636

iii) $E(Y/X)$

$$f_{Y/y} = \frac{f(x,y)}{f(x)}$$

$$f(x) = \frac{3}{6875} \int_{10}^{20} x^2 + xy \, dy$$

$$= \frac{3}{6875} (10x^2 + 150x)$$

$$= \frac{30x}{6875} (x + 15)$$

$$f_{y/x} = \frac{3}{6875} \frac{x(x+y)}{30(x)} \times \frac{6875}{(x+15)} = \frac{x+y}{10(x+15)}$$

$$E(Y/X) = \int_{10}^{20} \frac{(x+y)}{10(x+15)} dy$$

$$= \frac{1}{10(x+15)} \left[\frac{xy^2}{2} + \frac{y^3}{3} \right]_{10}^{20}$$

$$= \frac{1}{x+15} \left(15x + \frac{700}{3} \right)$$

$P(Y > \frac{1}{3}X + 10)$

$$f(y) = \frac{3}{6875} \int_0^5 x^2 + xy \, dx$$

$$= \frac{3}{6875} \left(\frac{x^3}{3} + \frac{x^2}{2} y \right)_0^5$$

$$= \frac{3}{6875} \left(\frac{125}{3} + \frac{25}{2} y \right)$$

So, $\frac{3}{6875}\left(\int_{\frac{1}{3}x+10}^{20} \frac{125}{3} + \frac{25}{2}y \, dy\right)$

$= \frac{3}{6875}\left(\frac{125y}{3} + \frac{25y^2}{4}\right)_{\frac{1}{3}x+10}^{20}$

$= \frac{3}{6875}\left(\frac{1250}{3} + 1875 - \frac{25x^2}{36} - \frac{125x}{7} - \frac{25x}{6}\right)$

6) $X \sim \text{Poisson}(\lambda)$ $\qquad$ $Y \sim \text{Poisson}(M)$

$M_X(t) = e^{\lambda(e^t - 1)}$ $\qquad$ $M_Y(t) = e^{M(e^t-1)}$

So, $Z = X - Y$

$M_Z(t) = E(e^{tZ}) = E(e^{t(X-Y)}) = E(e^{tX} \cdot e^{-tY})$

$= \frac{E(e^{tX})}{E(e^{tY})}$

$= \frac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}} = e^{(\lambda-\mu)(e^t-1)}$

So by uniqueness of MGF of $Z = e^{(\lambda-\mu)(e^t-1)} \sim \text{Poisson}(\lambda-\mu)$

now, let $Z = X+Y$

$E(e^{tZ}) = e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)}$

$= e^{(\lambda+\mu)(e^t-1)}$

So, by uniqueness $Z \sim \text{Poisson}(z+\mu)$

7)

			x		
		1	2	3	4
y	1	0.2	0	0.05	0.15
	2	0	0.3	0.1	0.2

i) $V(X/Y = 2) = E(X^2/Y = 2) - [E(X/Y-2)]^2$

$$E(X^2/Y = 2) = \frac{\sum x^2 \, P(X = x, y = 2)}{P(Y = 2)}$$

when $x = 1$ + when $x = 2$ + when $x = 3$ + when $x = 4$

$$\frac{1^2 \, P(X = 1, y = 2)}{(y = 2)} \text{ i.e } \frac{1(0)}{0.6} + \frac{4(0.3)}{0.6} + \frac{9(0.1)}{0.6} + \frac{16(0.2)}{6}$$

$$= 8.8333$$

$$E(X/Y = 2) = \sum_x \frac{P(X = x, y = 2)}{P(y = 2)}$$

$$= \frac{1(0)}{0.6} + \frac{2(0.3)}{0.6} + \frac{3(0.1)}{0.6} + \frac{4(0.2)}{0.6} = 2.8333$$

$$V(X/y = 2) = 8.333 - (2.8333)^2$$

$$= 0.80757$$

ii) $f(u,v) = \frac{48}{67}(2uv - u^2)$ $\quad 0 < u < 1, \; \frac{u}{2} < v < 2$

$E(u/v = v) = ?$

$$f\frac{(u,v)}{f(v)} \rightarrow f^{v}/_{v} = v$$

$$\therefore f(v) = \frac{48}{67} \int_0^1 (2uv - u^2)\, du$$

$$= \frac{48}{67}\left(u^2 v - \frac{u^3}{3}\right)_0^1$$

$= \frac{48}{67}(v - 1/3)$

$= \frac{16}{67}(3v - 1)$

$\frac{f(u,v)}{f(v)} = \frac{\cancel{48}^{3}}{\cancel{67}} \frac{(2uv^2 - u^2)}{(3v-1)} \times \frac{\cancel{67}}{\cancel{16}}$

$= \frac{3}{3v-1}(2uv^2 - u^2)$

$\therefore E(u/v = v) = \frac{3}{3v-1} \int_0^1 u(2uv^2 - u^2)\, du$

$= \frac{3}{3v-1}\left[\frac{2u^3v^2}{3} - \frac{u^4}{4}\right]_0^1$

$= \frac{8v^2 - 3}{4(3v-1)}$

8) $X \sim \text{Gamma}(3,2)$; $E(X) = 3/2$; $V(X) = 3/4$

$E(Y/X = x) = 3x+1$ $\quad$ $\text{var}(Y/X = x) = 2x^2 + 5$

$E(Y) = E(E(Y = x)) \rightarrow E(3x+1)$

$= 3(E(X)) + 1$

$= 3 \times 3/2 + 1$

$= 11/2$

$V(Y) = E[V(X/Y)] + V(E(X/Y))$

$= E(2x^2 + 5) + V(3x+1)$

$= 2E(x^2) + 5 + 9V(X)$

$\cancel{V(X)}$ $V(X) = E(X^2) - [E(X)]^2$

$E(X^2) = 3/4 + 9/4 = 3$

$\therefore V(Y) = 2(3) + 5 + 9(3/4)$

$= \frac{71}{4}$

9) $E(X) = 3$ $\quad$ $E(Y) = 4$

$V(X) = 4$ $\quad$ $V(Y) = 1$

Co-relation co-eff $= -0.3\sqrt{4} \rightarrow cov(x, y)$

$= -0.6$

$Z = x + y$

i) $cov(X, X + y) = cov(X, X) + cov(X, Y)$

$= V(X) + cov(X, Y)$

$= 4 + (-0.6)$

$= 3.4$

ii) $var(Z) = cov(X + Y, X + Y)$

$= cov(X, X) + 2 cov(X, X) + cov(Y, Y)$

$= V(X) + 2 cov(X, Y) + V(Y)$

$= 4 + 2(-0.6) + 1$

$= 5 - 1.2$

$= 3.8$

10) $f(x,y) = kx^{-\alpha} e^{-y/\beta}$

$$k\int_1^{\infty}\int_1^{\infty} x^{-\alpha} e^{-y/\beta}\,dx\,dy = 1$$

$$\therefore k\int_1^{\infty} e^{-y/\beta}\left[\frac{x^{-\alpha+1}}{1-\alpha}\right]_1^{\infty} = 1$$

$$\therefore \frac{k}{1-\alpha}\int_1^{\infty} e^{-y/\beta} = 1$$

$$\therefore \frac{k}{\alpha-1}\left(\frac{e^{-y/\beta}}{-1/\beta}\right)_1^{\infty} = 1$$

$$\therefore \frac{-k}{\alpha-1}(\beta)\,(e^{-1/\beta}) = 1$$

$$k = \frac{\alpha-1}{\beta\,(e^{-1/\beta})}$$