Assignment
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Q1

i)
$$\begin{array}{c|cc} & B & O \\ \hline B & -\lambda & \lambda \\ O & \mu & -\mu \end{array}$$

ii) $B \rightarrow \exp(\lambda_i)$

$O \rightarrow \exp(\mu_i)$

iii)
$$\frac{d}{dt}\, {}_tP_s^{BB} = {}_tP_s^{BB}(-\lambda_t) + {}_tP_s^{BO}(\mu_i)\, i$$
$$\frac{d}{dt}\, {}_tP_s^{BO} = {}_tP_s^{BO}(-\mu_t) + {}_tP_s^{BB}(\lambda_t)$$

iv ${}_tP_s^{BB} + {}_tP_s^{BO} = 1$

substitute (i)

$$\frac{d}{dt}\, {}_tP_s^{BB} = -\lambda_t P^{BB} + \mu(1 - {}_tP_s)^{BB}$$
$$\frac{d}{dt}\, {}_tP_s^{BB} = {}_tP_s^{BB}(\lambda + \mu) + \mu$$
$$\frac{d}{dt}\, {}_tP_s^{BB} + (\lambda + \mu)\, {}_tP_s^{BB} + \mu$$
$$IF = e^{\int(\lambda + \mu)dt}$$

$$\frac{d}{dt}\left[\exp((\lambda + \mu)t) \cdot {}_tP_s^{BB}\right] = \mu \exp((\lambda + \mu)t)$$
$$\exp((\lambda + \mu)t) \cdot {}_tP_s^{BB}$$
$$= \frac{\mu}{\lambda + \mu} \exp((\lambda + \mu)t) + C$$

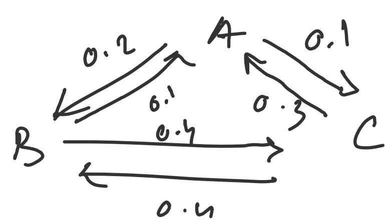
$$\text{Constant} = \frac{\lambda}{\lambda + \mu}$$

Constant $= \dfrac{\lambda}{\lambda+\mu}$

$${}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \exp\left((-\lambda+\mu)t\right)$$

Q2

$$\begin{array}{c|ccc} & A & B & C \\ \hline A & -0.3 & 0.2 & 0.1 \\ B & 0.1 & 0.5 & 0.4 \\ C & 0.3 & 0.4 & 0.4 \end{array}$$

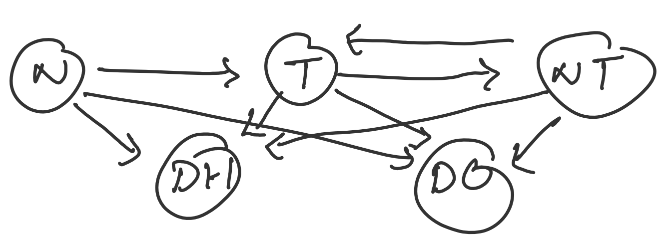


ii $\dfrac{d}{dt} P_{AA}(t) = {}_tP_s^{AA}(-0.3) + {}_tP_s^{AB}(0.2) + {}_tP_s^{A}(0.1)$

$$\frac{d}{dt} P_{AB}^{t} = {}_tP_s^{AB}(0.1) \cdot {}_tP_s^{BB}(-0.5)$$
$$+ {}_tP_s^{AA}(-0.3) \; {}_tP_s^{AB}(0.1)$$
$$+ {}_tP_s^{AC}(0.1) \; {}_tP_s^{CB}(0.4)$$

Q3

i)



$$\frac{d}{dt}\left({}_tP_x^{34}\right) = {}_tP_x^{32}\,\mu_{x+t}^{24} + {}_tP_x^{33}\,\mu_{x+t}^{34}$$

Proof: By markov assumption

$$ {}_{dt+t}p_x^{34} = {}_tp_x^{31}\,{}_{dt}p_x^{14} + {}_tp_x^{32}\,{}_{dt}p_{x+t}^{24} + {}_tp_x^{33}\,{}_{dt}p_{x+t}^{34} + {}_tp_x^{34}\,{}_{dt}p_{x+t}^{44} + {}_tp_x^{35}\,{}_{dt}p_{x+t}^{54} $$

as $\quad {}_{dt}p_{x+t}^{54} = {}_tp_x^{31} = 0$

$$ {}_{dt+t}p_x^{34} = {}_tp_x^{32}\,{}_{dt}p_{x+t} + {}_tp_x^{33}\cdot p_{x+t} + {}_tp_x^{34}\,{}_{dt}p_{x+t}^{44} $$

Given that ${}_{dt}p_{x+t}^{44} = 1$

assuming that for small dt

$$ {}_{dt}p_{x+t}^{ij} = \mu_{x+t}^{ij}\,dt + o(dt) $$

where $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$

then $\quad {}_{dt+t}p_x^{34} = {}_tp_x^{32}\,\mu_{x+t}^{24}\,dt + {}_tp_x^{33}\,\mu_{x+t}^{34}\,dt + {}_tp_x^{34} + o(dt)$

$\therefore \quad {}_{dt+t}p_x^{34} - {}_tp_x^{34} = {}_tp_x^{32}\,\mu_{x+t}^{24}\,dt + {}_tp_x^{33}\,\mu_{x+t}^{34}\,dt + o(dt)$

h) i) As mean is equal to parameter then are 3calls per hour.

ii) the process is memoryless so the fact that Fred has not called for 15minutes is irrelevant so the next call is in 20 min.

iii) The prob of zero calls in 0.5 hrs.

$$ P_0(0.5) = \frac{e^{-1.5}(1.5)^0}{0!} $$

$$ = \underline{\underline{0.2231}} $$

iv) The expected time that Fred is on phone is expected number of calls times the expected length of a call $= 3 \times 7 = 21$

probability that phone is engaged is $21/60 = 0.35$

times ... expected length of a call ...

probability that phone is engaged is $21/60 = 0.35$

5i) $P_{HH}(x, t+dt) = P_{HH}(x,t)\, P_{HH}(t, t+dt) + P_{HS}(x,t)\, P_{SH}(t, t+dt) + P_{HD}(x,t)\, P_{DH}(t, t+dt)$

$$P_{HH}(x, t+dt) = P_{HH}(x,t)\, P_{HH}(t, t+dt) + P_{HS}(x,t)\, P_{SH}(t, t+dt)$$

Assuming for small dt

$$P_{ij}(t, t+dt) = \lambda_{ij}(t)\,dt + o(dt) \qquad i \neq j$$

$$P_{ii}(t, t+dt) = 1 + \lambda_{ii}(t)\,dt + o(dt)$$
$$= 1 - \sum_{j \neq 1} \lambda_{ij}(t)\,dt + o(dt)$$

where the λ are the instantaneous transition rates and $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$

then $P_{HH}(x,t)(1 - \sigma(t)\,dt - \mu(t)\,dt) + P_{HS}(x,t)\, p_{HS}(x,t)\, \rho(t, C_t) + o(dt)$

$$P_{HH}(x, t+dt) - P_{HH}(x,t) = P_{HH}(x,t)\,(-\sigma(t) - \mu(t))\,dt + P_{HS}(x,t)\, \rho(t, C_t)\,dt + o(dt)$$

$$\frac{d}{dt} P_{HH}(x,t) = \lim_{dt \to 0} \frac{P_{HH}(x, t+dt) - P_{HH}(x,t)}{dt}$$
$$= P_{HH}(x,t)\,(-\sigma(t) - \mu(t)) + P_{HS}(x,t)\, \rho(t, C_t)$$

ii) $\frac{d}{dt} P_{\overline{HH}}(0,t) = -(\sigma(t) + \mu(t))\, P_{\overline{HH}}(t)$

$$\frac{1}{P_{\overline{HH}}(0,t)} \frac{d}{dt} P_{\overline{HH}}(0,t) = -(\sigma(t) + \mu(t)) = \frac{d}{dt} \ln P_{\overline{HH}}(t).$$

Integrating BS

$$\left[\ln P_{\overline{HH}}(0,t)\right]^t = \int^t \left[-(\sigma(s) + \mu(s))\right] ds$$

integrating gives

$$[\ln P_{\bar{H}\bar{H}}(0,t)]_0^t = \int_{s=0}^{t} -(\sigma(s) + \mu(s))\, ds$$

$P_{\bar{H}\bar{H}}(0) = 1$

$$P_{\bar{H}\bar{H}}(0,t) = e^{-\int_0^t \sigma(s) + \mu(s)\, ds}$$

6) • All 3 processes have a discrete state space
• A markov chain & markov jump chain both operate in discrete time but markov jump process operates in continous time.
• All have the markov property which is either that the future development of the process can be predicted from its present

7) the MLE of transition intensities from state i to state j is the number of transaction from state i to j divided by the total waiting time in state i.
To estimate the transaction intensity exactly we need the total time spent in each state or the entry and exit times for each individual for each state, and the total number of transactions of each type made.

ii) define $P_{AA}(s,t)$ to be the probability of being in state Active at time s+t if if active at time s.
then either $\frac{\partial}{\partial t} P_{AA}(s,t) = -P_{AA}(s,t)\mu$

$$\frac{\partial}{\partial t} P_{AT}(s,t) = P_{AA}(s,t)\mu$$

iii) $\frac{1}{P_{AA}(t)} \frac{\partial}{\partial t} P_{AA}(t) = -\mu$

$$\frac{\partial}{\partial t}(\ln(P_{AA}(t))) = -\mu$$

hence $P_{AA}(t) = \exp\{-\mu t + C\}$

As $P_{AA}(0) = 1$ $C = 0$ so

$P_{AA}(t) = \exp(-\mu t)$

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A claim occurs with cost £C if moves to state 'Theft claim'.
Thus expected cost is $C(1-\exp(-\mu T))$

iv

$$\begin{tikzcd} \text{Active} \arrow[r, "\mu"] \arrow[d, "\lambda"'] & \text{Theft claim} \\ \text{Lapsed} & \end{tikzcd}$$

v we now have $\frac{\partial}{\partial t} P_{AA}(t) = -P_{AA}(t)(\mu+\lambda)$

So $P_{AA}(t) = \exp(-(\mu+\lambda)t)$

we want $\frac{\partial}{\partial t} P_{AT}(t)\mu = \mu\exp(-(\mu+\lambda)t)$

$$P_{AT}(t) = \left| \frac{-\mu}{(\mu+\lambda)} \exp(-(\mu+\lambda)t) \right|_0^T = \frac{\mu}{\mu+\lambda}\left(1-\exp(-(\mu+\lambda)T)\right)$$

So claims become $\frac{\mu}{\mu+\lambda} C(1-\exp(-(\mu+\lambda)T))$

8 i) $\{0, 1, 2, 3, 4 \dots\}$

ii)

$$\begin{tikzcd} 0 & 1 \arrow[l, "\mu"] \arrow[r, shift left, "\lambda"] & 2 \arrow[l, shift left, "\mu"] \arrow[r, shift left, "\lambda"] & 3 \arrow[l, shift left, "\mu"] \arrow[r, shift left, "\lambda"] & 4 \arrow[l, shift left, "\mu"] \arrow[r, shift left, dashed, "\lambda"] & \cdots \arrow[l, shift left, dashed, "\mu"] \end{tikzcd}$$

iii) Generator matrix

0 1 2 3 4

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ \mu & -(\mu+\lambda) & \lambda & 0 & 0 \\ 0 & \mu & -(\mu+\lambda) & \lambda & 0 \\ 0 & 0 & \mu & -(\mu+\lambda) & \lambda \\ 0 & 0 & 0 & \mu & -(\mu+\lambda) \end{bmatrix}$$

iv A jump chain is each distinct state visitted in the order visited where the time set is the times when states are moved between

v 0

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 \\ \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 & 0 \\ 0 & \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) & 0 \\ 0 & 0 & \mu/(\mu+\lambda) & 0 & \lambda/(\mu+\lambda) \\ 0 & 0 & 0 & \mu/(\mu+\lambda) & 0 \end{matrix}$$

vi) $\left(\frac{\mu}{\mu+\lambda}\right)^3$

10 i) $\frac{d}{dt} P_{\overline{AA}}(t) = -2t \, P_{\overline{AA}}(t)$

$\Rightarrow \frac{d}{dt}\left[\ln P_{\overline{AA}}(t)\right] = -2t$

$\ln P_{\overline{AA}}(s) = -s^2 + \text{constant}$

WKT $P_{\overline{AA}}(0) = 1$ hence constant $= 0$

hence $P_{\overline{AA}}(s) = \exp^{-s^2}$

ii) P(in first to visit at time T in state A at $t=0$)

$= \int_0^T$ P(remains in A to time s)

$\times$ P(transition to B in time s, s+ds)
$\times$ P(Remains in B to time T) ds

× P(transition to B in time s, s+ds)
× P(Remains in B to time T) ds

$$= \int_0^T P_{\overline{AA}}(s) \times 2s \times P_{\overline{BB}}(s,T)\,ds$$

from (i)

$$= \int_0^T e^{-s^2} \times 2s \times e^{-T^2+s^2}\,ds$$

$$= \int_0^T 2s \times e^{-T^2}\,ds$$

$$= e^{-T^2} \times T^2$$

iii a)

b) • Initially probability increases from 0 to T = 0 & accelerates as the transition rate from A to B increases.
• However, as transitions increase, it becomes more likely that the process has already visited state B and jumped back to A therefore the probability of being in the first visit to B tends to 0.

c) differentiate to find turning point

$$\frac{d}{dt}\left[e^{-t^2} \times t^2\right] = 2t \times e^{-t^2} - 2t^3 \times e^{-t^2}$$

set derivative equal to zero

$$e^{-t^2} \times 2t(1-t^2) = 0$$

ie $t = 1$ for positive solution

ie $f=1$ for Positive solution
and from above it is maximum.

11 i) let N denote the number of claims up to time. Since the poisson process has stationary increments, we many take $t=0$, so that the required conditional distribution is

$$P(T_0 \le y \mid N_s = 1) = \frac{P(T_0 \le y, N_s = 1)}{P(N_s=1)}$$

$$= \frac{P(N_y = 1, N_s - N_y = 0)}{P(N_s = 1)}$$

But $N_s - N_y$ is independent of N_y and has same distribution as N_{s-y}

$$\therefore R.H.S = \frac{\lambda y e^{-\lambda y} \cdot e^{-\lambda(s-y)}}{\lambda s e^{-\lambda s}} = \frac{y}{s}$$

ie the cdf of uniform dis on $[0,1]$

ii) Since holding time are independent each has a exponential dist & Jd is

$$\lambda^n e^{-\lambda(t_1+t_2+\dots+t_n)} 1_{\{t_1, t_2, \dots t_n > 0\}}$$

iii) from i)

$$P(N_s = k \mid N_t = n) = \frac{P(N_s = k, N_t = n)}{P(N_t = n)}$$

$$= \frac{P(N_s = k, N_t - N_s = n-k)}{P(N_t = n)}$$

using again that Poi Process has stationary & independent increments, and the number of claims in an interval $[0,t]$ is poisson(t),

$$P(N_s = k \mid N_t = n) = \frac{\frac{e^{-\lambda s} \cdot (\lambda s)^k}{k!} \cdot \frac{e^{-\lambda(t-s)} \cdot \lambda^{n-k} (t-s)^{n-k}}{(n-k)!}}{e^{-\lambda t} (\lambda t)^n}$$

$$\frac{\dfrac{\quad}{k!} \cdot \dfrac{\quad}{(n-k)!}}{\dfrac{e^{-\lambda t} \cdot (\lambda t)^n}{n!}}$$

$$= \frac{n!}{k!(n-k)!} \cdot \frac{s^k (t-s)^{n-k}}{t^k t^{n-k}}$$

$$= \binom{n}{k} \left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}$$

Which is binomial with parameter n, s/t.