

1)

i) Arbitrage opportunity is a situation where we can make a certain profit with no risk. This is sometimes described as a free lunch.

An arbitrage opportunity means that:

a) we can start at time 0 with a portfolio that has a net value of zero (implying that we are long in some assets and short in others). This is usually called a zero-cost portfolio.

b) at some future time, T:

the probability of a loss is 0 and the probability that we make a strictly positive profit is greater than 0.

If such an opportunity existed then we could multiply up this portfolio as much as we wanted to make as large a profit as we desired.

ii) The Law of one price states that any two portfolios that behave in exactly the same way must have the same price. If this were not true, we could buy the 'cheap' one and sell the 'expensive' one to make an arbitrage (risk-free) profit.

iii)

a) Using Put- Call parity, the value of put option should be:

$$\begin{aligned} p_t &= c_t + K \exp(-r(T-t)) - S \exp(-q(T-t)) \\ &= 30 + 120 \exp(-.05 * .25) - 125 \exp(-.15 * .25) \\ &= 28.11 \end{aligned}$$

b) If the put options are only Rs. 23 then they are cheap. If things are cheap then we buy them.

So, looking at the put-call parity relationship, we "buy the cheap side and sell the expensive side", i.e., we buy put options and shares and sell call options and cash.

For example:

- sell 1 call option Rs. 30
- buy 1 put option (Rs. 23)
- buy 1 share (Rs. 125)
- sell (borrow) cash Rs. 118

This is a zero-cost portfolio and, because put-call parity does not hold, we know it will make an arbitrage profit. We can check as follows:

In 3 months', time, repaying the cash will cost us:

$$118 \exp(0.05 * 3 / 12) = \text{Rs. } 119.48$$

We also receive dividends  $d$  on the share.

If the share price is above 120 in 3 months' time, then the other party will exercise their call option and we will have to give them the share. They will pay 120 for it and our profit is:

$$120 - 119.48 + d = 0.52 + d$$

(The put option is useless to us)

If the share price is below 120 in 3 months' time, then we will exercise our put option and sell it for 120. Our profit is:

$$120 - 119.48 + d = 0.52 + d$$

(The call option is useless to the other party and will expire worthless)

$$2) \quad \text{Let } dX_t = A_t dt + B_t dZ_t,$$

$$\text{Where, } A_t = \alpha\mu(T-t), B_t = \sigma\sqrt{(T-t)} \quad \text{Eq 1}$$

$$dF = \frac{\partial f}{\partial x} B_t dZ_t + \left( \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} A_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} B_t^2 \right) dt \quad (\text{Ito's Lemma})$$

$$dF = -f B_t dZ_t + f \frac{\partial m}{\partial t} (T-t) dt - f A_t dt + \frac{1}{2} f B_t^2 dt$$

$$\left[ \text{Since } \frac{\partial f}{\partial x} = -e^{m(T-t)-x} \text{ and } \frac{\partial f}{\partial x} \right. \\ \left. = \frac{\partial m}{\partial t} (T-t) * e^{m(T-t)-x} \text{ (using chain rule) and } \frac{\partial^2 f}{\partial x^2} \right. \\ \left. = e^{m(T-t)-x} \right]$$

$$dF = f \left( \frac{\partial m}{\partial t} (T-t) - A_t + \frac{1}{2} B_t^2 \right) dt - f B_t dZ_t$$

$$\text{For } f \text{ to be a martingale, } \frac{\partial m}{\partial t} (T-t) - A_t + \frac{1}{2} B_t^2 = 0$$

$$\text{Thus, } \frac{\partial m}{\partial t} (T-t) = A_t - \frac{1}{2} B_t^2$$

$$\text{Substituting Eq 1 above gives } \frac{\partial m}{\partial t} = \alpha\mu - \frac{1}{2} \sigma^2$$

3)

i) Given

$$F(t, x) = e^{-t} x^2$$

$$\frac{df}{dt} = -e^{-t} x^2 = -f$$

$$\frac{df}{dx} = 2e^{-t} x$$

$$\frac{df}{dx^2} = 2e^{-t}$$

$$Y_t = f(t, X_t)$$

Applying Ito's Lemma

$$dY_t = \frac{df}{dt} dt + \frac{df}{dx} dX_t + \frac{1}{2} \frac{d^2 f}{dx^2} \sigma^2 X_t^2 dt$$

$$dY_t = -f dt + 2e^{-t} X_t dX_t + e^{-t} \sigma^2 X_t^2 dt$$

$$dY_t = -Y_t dt + 2e^{-t} X_t^2 \frac{dX_t}{X_t} + e^{-t} \sigma^2 X_t^2 dt$$

$$-Y_t dt + 2Y_t [0.24 dt + \sigma^2 Y_t dt]$$

$$\frac{dY_t}{Y_t} = [2 * (0.25) - 1 + \sigma^2] dt + 2\sigma dW_t$$

$$\frac{dY_t}{Y_t} = [\sigma^2 - 0.5] Y_t dt + 2\sigma Y_t dW_t$$

- ii) The process is martingale if drift is zero. This means  
 $\sigma^2 - 0.5 = 0$   
 i.e.  $\sigma^2 = 0.5$

4) Let  $n$  ex-dividend dates are anticipated for a stock and  $t_1 < t_2 < \dots < t_n$  are the times before which the stock goes ex-dividend. Dividends are denoted by  $d_1, \dots, d_n$ .

If the option is exercised prior to the ex-dividend date then the investor receives  $S(t_n) - K$ . If the option is not exercised, the price drops to  $S(t_n) - d_n$ .

The value of the American option is greater than  $S(t_n) - d_n - K \exp(-r(T-t_n))$ .

It is never optimal to exercise the option if  $S(t_n) - d_n - K \exp(-r(T-t_n)) \geq S(t_n) - K$  i.e.,  $d_n \leq K * (1 - \exp(-r(T-t_n)))$ .

Using this equation: we have  $K * (1 - \exp(-r(T-t_n))) = 350 * (1 - \exp(-0.95 * (0.8333 - 0.25))) = 18.87$  and  $65 * (1 - \exp(-0.95 * (0.8333 - 0.25))) = 10.91$ . Hence it is never optimal to exercise the American option on the two ex-dividend rates.

5) The required probability is the probability of the stock price being greater than Rs. 258 in 6 months' time.

The stock price follows Geometric Brownian motion i.e.  $S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$

Therefore  $\ln(S_t)$  follows normal distribution with mean  $\ln(S_0) + \left(\mu - \frac{\sigma^2}{2}\right)t$  and variance  $\sigma^2 t$

Implies  $\ln(S_t)$  follows  $\varphi\left(\ln 254 + \left(0.16 - \frac{0.35^2}{2}\right) * 0.5, 0.35 * 0.5^{\frac{1}{2}}\right) = \varphi(0.59, 0.247)$

This means  $[\ln(S_t) - \ln(S_0)] / (\sigma \sqrt{t})$  follows standard normal distribution. Hence the probability that stock price will be higher than the strike price of Rs. 258 in 6 months' time =  $1 - N(5.55 - 5.59) / 0.247 = 1 - N(-0.1364) = 0.5542$ .

The put option is exercised if the stock price is less than Rs. 258 in 6 months' time. The probability of this =  $1 - 0.5542 = 0.4457$

6)

i) The given relationship can be written as:

$$S_t = S_0 e^{\mu t + \sigma B_t}$$

Since  $S_t$  is a function of standard Brownian motion,  $B_t$ , applying Ito's Lemma, the SDE for the underlying stochastic process becomes:

$$dB_t = 0 \times dt + 1 \times dB_t$$

Let  $G(t, B_t) = S_t = S_0 e^{\mu t + \sigma B_t}$ , then

$$\frac{dG}{dt} = \mu S_0 e^{\mu t + \sigma B_t} = \mu S_t$$

$$\frac{dG}{dB_t} = \sigma S_0 e^{\mu t + \sigma B_t} = \sigma S_t$$

$$\frac{d^2 G}{dB_t^2} = \sigma^2 S_0 e^{\mu t + \sigma B_t} = \sigma^2 S_t$$

Hence, using Ito's Lemma from Page 46 in the Tables we have:

$$dG = [0 \times \sigma S_t + \frac{1}{2} \times 1^2 \times \sigma^2 S_t + \mu S_t]dt + 1 \times \sigma S_t dB_t$$

$$\text{i.e. } dS_t = (\mu + \frac{1}{2} \sigma^2) S_t dt + \sigma S_t dB_t$$

Thus,

$$\frac{dS_t}{S_t} = \sigma dB_t + (\mu + \frac{1}{2} \sigma^2) dt$$

So,  $c_1 = \sigma$  and  $c_2 = \mu + \frac{1}{2} \sigma^2$

ii) The expected value of  $S_t$  is:

$$E[S_t] = E[S_0 e^{\mu t + \sigma B_t}] = S_0 e^{\mu t} E[e^{\sigma B_t}]$$

Since  $B_t \sim N(0,1)$ , its MGF is  $E[e^{\theta B_t}] = e^{\frac{1}{2}\theta^2 t}$

So,  $E[S_t] = S_0 e^{\mu t} \times e^{\frac{1}{2}\sigma^2 t} = S_0 e^{\mu t + \frac{1}{2}\sigma^2 t}$

The variance of  $S_t$  is:

$$\begin{aligned} \text{Var}[S_t] &= E[S_t^2] - (E[S_t])^2 = E[S_0^2 e^{2\mu t + 2\sigma B_t}] - (S_0 e^{\mu t + \frac{1}{2}\sigma^2 t})^2 \\ &= S_0^2 e^{2\mu t} E[e^{2\sigma B_t}] - S_0^2 e^{2\mu t} + \sigma^2 t \\ &= S_0^2 e^{2\mu t + 2\sigma^2 t} - S_0^2 e^{2\mu t + \sigma^2 t} = S_0^2 e^{2\mu t} (e^{2\sigma^2 t} - e^{\sigma^2 t}) \end{aligned}$$

iii)  $\text{Cov}[S_{t_1}, S_{t_2}] = E[S_{t_1}, S_{t_2}] - E[S_{t_1}]E[S_{t_2}]$

From above,

$$E[S_{t_1}] = S_0 e^{\mu t_1 + \frac{1}{2}\sigma^2 t_1} \text{ and } E[S_{t_2}] = S_0 e^{\mu t_2 + \frac{1}{2}\sigma^2 t_2}$$

The expected value of the product is:

$$\begin{aligned} E[S_{t_1}, S_{t_2}] &= E[S_0 \exp(\mu t_1 + \sigma B_{t_1}) S_0 \exp(\mu t_2 + \sigma B_{t_2})] \\ &= S_0^2 e^{\mu(t_1 + t_2)} E[\exp(\sigma B_{t_1} + \sigma B_{t_2})] \end{aligned}$$

To evaluate this we need to split  $B_{t_2}$  into two independent components:

$$B_{t_2} = B_{t_1} + (B_{t_2} - B_{t_1}) \text{ where } B_{t_2} - B_{t_1} \sim N(0, t_2 - t_1)$$

Hence,

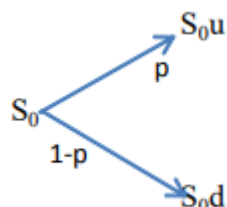
$$\begin{aligned} E[S_{t_1}, S_{t_2}] &= S_0^2 e^{\mu(t_1 + t_2)} E[\exp(\sigma B_{t_1} + \sigma \{B_{t_1} + (B_{t_2} - B_{t_1})\})] \\ &= S_0^2 e^{\mu(t_1 + t_2)} E[\exp(2\sigma B_{t_1} + \sigma \{B_{t_2} - B_{t_1}\})] \\ &= S_0^2 e^{\mu(t_1 + t_2)} E[\exp(2\sigma B_{t_1})] E[\exp\{\sigma \{B_{t_2} - B_{t_1}\}\}] \\ &= S_0^2 e^{\mu(t_1 + t_2)} \exp(2\sigma^2 t_1) \exp\left[\frac{1}{2}\sigma^2 (t_2 - t_1)\right] \\ &= S_0^2 e^{\mu(t_1 + t_2)} \exp\left(\frac{3}{2}\sigma^2 t_1 + \frac{1}{2}\sigma^2 t_2\right) \end{aligned}$$

Putting all the equations together:

$$\begin{aligned} \text{Cov}[S_{t_1}, S_{t_2}] &= S_0^2 e^{\mu(t_1 + t_2)} \exp\left(\frac{3}{2}\sigma^2 t_1 + \frac{1}{2}\sigma^2 t_2\right) - S_0 e^{\mu t_1 + \frac{1}{2}\sigma^2 t_1} \cdot S_0 e^{\mu t_2 + \frac{1}{2}\sigma^2 t_2} \\ &= S_0^2 e^{\mu(t_1 + t_2)} \left(\exp\left(\frac{3}{2}\sigma^2 t_1\right) - \exp\left(\frac{1}{2}\sigma^2 t_1\right)\right) \exp\left(\frac{1}{2}\sigma^2 t_2\right) \end{aligned}$$

7)

- i) Setting up the commodity tree using  $u$  for up move and  $d$  for down move,  $p$  is up-step probability:



Where  $p$  is the up probability and  $(1-p)$  the down probability.

Then  $E(C_t) = S_0[p u + (1-p)d]$ , and

$$\begin{aligned}\text{Var}(C_t) &= E(C_t^2) - E(C_t)^2 \\ &= S_0^2 [p u^2 + (1-p)d^2] - S_0^2 [p u + (1-p)d]^2 \\ &= S_0^2 [p u^2 + (1-p)d^2 - (p u + (1-p)d)^2] \\ &= S_0^2 [p(1-p)u^2 + p(1-p)d^2 - 2p(1-p)] \quad (\because d = 1/u) \\ &= S_0^2 p(1-p)(u-d)^2\end{aligned}$$

Equating moments:

$$S_0 e^{rt} = S_0 [p u + (1-p)d] \quad \text{_____ (A)}$$

$$\text{And } \sigma^2 S_0^2 t = S_0^2 p(1-p)(u-d)^2 \quad \text{_____ (B)}$$

From (A) we get

$$p = \frac{e^{rt} - d}{u - d} \quad \text{_____ (C)}$$

Substituting p into equation (B), we get

$$\begin{aligned}\sigma^2 t &= \frac{e^{rt} - d}{u - d} \left(1 - \frac{e^{rt} - d}{u - d}\right) (u - d)^2 \\ &= - (e^{rt} - d)(e^{rt} - u) = (u + d) e^{rt} - (1 + e^{2rt})\end{aligned}$$

Putting  $d = 1/u$ , and multiplying through by u we get

$$u^2 e^{rt} - u(1 + e^{2rt} + \sigma^2 t) + e^{rt} = 0$$

This is a quadratic in u which can be solved in the usual way.

ii)

$$\begin{aligned}\text{a) } \sigma &= 0.15, t = 0.25 \Rightarrow u = \exp(.15 \cdot \sqrt{.25}) = \exp(.075) = \\ &1.077884, d = 1/u = .927774\end{aligned}$$

The tree is

| t=0 | t=.25  | t=.5   | t=.75   |        |
|-----|--------|--------|---------|--------|
|     |        |        | 100.186 | Node A |
|     |        | 92.947 |         |        |
|     | 86.231 |        | 86.232  | Node B |
| 80  |        | 80.001 |         |        |
|     | 74.22  |        | 74.22   | Node C |
|     |        | 68.857 |         |        |
|     |        |        | 63.882  | Node D |

b)  $r = 0$ , we have  $p = \frac{e^{rt}-d}{u-d} = \frac{(1-.927744)}{1.077884-.927744} = .48126$

Discounting back the final payoff at  $t=.75$  to  $t=0$  along the tree using  $p$  and  $(1-p)$ , we get

| t=0   | t=.25 | t=.5   | t=.75  |        |
|-------|-------|--------|--------|--------|
|       |       |        | 20.186 | Node A |
|       |       | 12.948 |        |        |
|       | 7.787 |        | 6.232  | Node B |
| 4.496 |       | 2.999  |        |        |
|       | 1.443 |        | 0      | Node C |
|       |       | 0      |        |        |
|       |       |        | 0      | Node D |

Hence value of the call option is 4.496.

c) The lookback call pays the difference between the minimum value and the final value.

Notate paths by U for up and D for down, in order

We get the payoffs

|     |                             |        |
|-----|-----------------------------|--------|
| UUU | $(100.186 - 80) = 20.186$   | Node A |
| UDU | $(86.232 - 80) = 6.232$     | Node B |
| UUD | $(86.232 - 80) = 6.232$     | Node B |
| UDD | $(74.22 - 74.22) = 0$       | Node C |
| DUU | $(86.232 - 74.22) = 12.012$ | Node B |
| DUD | $(74.22 - 74.22) = 0$       | Node C |
| DDU | $(74.22 - 68.857) = 5.363$  | Node C |
| DDD | $(63.882 - 63.882) = 0$     | Node D |

The lookback payoffs are, for each successful path (i.e., with a non- zero result)

Probabilities of arriving at each node are:

Node A =  $p^3 = .11147$

Node B =  $p^2(1-p) = .12015$

Node C =  $p(1-p)^2 = .12950$

Node D =  $p(1-p)^3 = .13959$

Hence the tree value of lookback option is:

$$(.11147 * 20.186) + (.12015 * [6.232 + 6.232 + 12.012]) + (.12950 * 5.363) = 5.8854$$

8)

- i) Consider a stock whose current price is  $S_0$  and an option whose current price is  $f$ . We suppose that the option lasts for time  $T$  and that during the life of the option the stock price can either move up from  $S_0$  to a new level  $S_{0u}$  or move down to  $S_{0d}$  where  $u > 1$  and  $d < 1$ .

Let the payoff be  $f_u$  if the stock price becomes  $S_{0u}$  and  $f_d$  if stock price becomes  $S_{0d}$ .

Let us construct a portfolio which consists of a short position in the option and a long position in  $\Delta$  shares. We calculate the value of  $\Delta$  that makes the portfolio risk-free.

Now if there is an upward movement in the stock, the value of the portfolio becomes  $\Delta S_{0u} - f_u$  and if there is a downward movement of stock, the value of the portfolio becomes  $\Delta S_{0d} - f_d$

The two portfolios are equal if  $\Delta S_{0u} - f_u = \Delta S_{0d} - f_d$

Or  $\Delta = (f_u - f_d) / (S_{0u} - S_{0d})$  so that the portfolio is risk-free and hence must earn the risk-free rate of interest.

This means the present value of such a portfolio is  $(\Delta S_{0u} - f_u) \exp(-rT)$

Where  $r$  is the risk-free rate of interest.

The cost of the portfolio is  $\Delta S_0 - f$

Since the portfolio grows at a risk-free rate, it follows that

$$(\Delta S_{0u} - f_u) \exp(-rT) = \Delta S_0 - f$$

$$\text{or } f = \Delta S_0 - (\Delta S_{0u} - f_u) \exp(-rT)$$

Substituting  $\Delta$  from the earlier equation simplifies to:

$$f = e^{-rT} [p f_u + (1 - p) f_d] \text{ where } p = \frac{[e^{rT} - d]}{u - d}$$

- ii) The option pricing formula does not involve probabilities of stock going up or down although it is natural to assume that the probability of an upward movement in

stock increases the value of call option and the value of put option decreases when the probability of stock price goes down

This is because we are calculating the value of option not in absolute terms but in terms of the value of the underlying stock where the probabilities of future movements (up and down) in the stock already incorporates in the price of the stock. However, it is natural to interpret  $p$  as the probability of an up movement in the stock price. The variable  $1-p$  is then the probability of a down movement such that the above equation can be interpreted as that the value of option today is the expected future value discounted at the risk-free rate

iii) The expected stock price  $E(S_T)$  at time  $T = pS_{0u} + (1-p)S_{0d}$

$$\text{or } E(S_T) = p S_0(u-d) + S_0d$$

Substituting  $p$  from above equation in (i) i.e.,  $p = \frac{e^{rT} - d}{u - d}$

$$\text{We get } E(S_T) = e^{rT} S_0$$

i.e., stock price grows at a risk-free rate or return on a stock is risk free rate

iv) In a risk neutral world individuals do not require compensation for risk or they are indifferent to risk. Hence expected return on all securities and options is the risk free interest rate. Hence value of an option is its expected payoff in a risk neutral discounted at risk free rate.

9)

i) The forward price is given by  $F = S \cdot \exp(rt)$  where  $S$  is the stock price,  $t$  is the delivery time and  $r$  is the continuously compounded risk-free rate of interest applicable up to time  $t$ .

Put-call parity states that:  $c + K \cdot \exp(-rt) = p + S$  where  $c$  and  $p$  are the prices of a European call and put option respectively with strike  $K$  and time to expiry  $t$  and  $S$  is the current stock price.

To compute  $F$ , we need to find  $S$  and  $r$ .  $t$  is given to be 0.25 years.

Substituting the values from the first two rows of the table in the put-call parity, we get two equations in two unknowns (S and r):

$$13.334 + 70 \cdot \exp(-0.25r) = 0.120 + S$$

$$8.869 + 75 \cdot \exp(-0.25r) = 0.568 + S$$

Solving the simultaneous equations for S and r, we get:

$$S = 82 \text{ and } r = 7\%$$

$$\text{Therefore, we get the forward price } F = 82 \cdot \exp(0.07 \cdot 0.25) = 83.45$$

- ii) Let the (continuously compounded, annualized) rate of interest over the next k months be  $r_k$ . Then the required forward rate  $r^F$  can be found from:

$$\exp(r_6 \cdot 0.5) = \exp(r_3 \cdot 0.25) \cdot \exp(r^F \cdot 0.25) \text{ or } 2 \cdot r_6 = r_3 + r^F$$

We know that  $r_3 = 7\%$ .

To find  $r_6$ , we substitute values from the last row in the put-call parity relationship and  $S = 82$ :

$$2.569 + 90 \cdot \exp(-0.5 \cdot r_6) = 7.909 + 82$$

Therefore,  $r_6 = 6\%$  and  $r^F = 5\%$

- iii) Using the put-call parity for each row in the given table, we get:

$$6.899 + a \cdot \exp(-0.07 \cdot 0.25) = 1.055 + 82$$

$$b + 80 \cdot \exp(-0.07 \cdot 0.25) = 1.789 + 82$$

$$2.594 + 85 \cdot \exp(-0.07 \cdot 0.25) = c + 82$$

Solving individually, we get:

$$a = 77.5 \quad b = 5.177 \quad c = 4.119$$

10)

- i) Since interest rates are assumed zero, the risk-neutral up-step probability is given as:

$$q = (1 - d) / (u - d)$$

where u and d are the sizes of up-step and down-step respectively

For a recombining tree,  $d = 1/u$ .

Substituting  $d = 1/u$  in the expression for q and simplifying, we get:

$$q = (1 - 1/u) / (u - 1/u) = 1 / (u + 1)$$

For no-arbitrage to hold, we must have  $u > 1 > d$ .

Then,  $u > 1 \Rightarrow u + 1 > 2 \Rightarrow q = 1 / (u + 1) < 1/2$ . Hence proved.

- ii) Since each step is one month and the expiry of the derivative is one year from now. Therefore, a 12-step recombining binomial tree needs to be created, i.e.  $n = 12$ . Further, at time  $T = 12$  months, the stock price will be  $S_{0u}^k d^{n-k}$  with risk-neutral probability  $\binom{n}{k} q^k (1-q)^{n-k}$  where  $q$ , the up-step probability is  $1/3$ ,  $u$ , the up-step size is  $2$ , and  $d = 1/u = 1/2$ .

We know that the derivative has a payoff  $\sqrt{\frac{S_T}{S_0}}$  at time  $T = 12$  months.

Thus, the current price of that derivative is:  $P = \sum_{k=0}^n \sqrt{\frac{S_T}{S_0}} \cdot \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k}$

Therefore,  $P = \sum_{k=0}^n \sqrt{\frac{S_0 u^k d^{n-k}}{S_0}} \cdot \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k} = \sum_{k=0}^n \sqrt{u^k d^{n-k}} \cdot \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k}$

$$P = \sum_{k=0}^n u^{\frac{k}{2}} d^{\frac{n-k}{2}} \cdot \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k} = \sum_{k=0}^n 2^{\frac{k}{2}} \left(\frac{1}{2}\right)^{\frac{n-k}{2}} \cdot \frac{n!}{k!(n-k)!} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k}$$

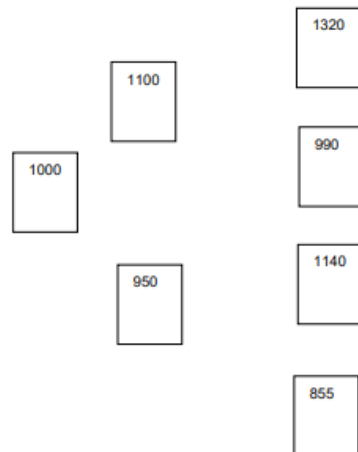
$$P = \sum_{k=0}^n 2^{\frac{k}{2}} \cdot \frac{n!}{k!(n-k)!} \frac{2^{n-k}}{3^n} = \sum_{k=0}^n 2^{\frac{n}{2}} \cdot \frac{n!}{k!(n-k)!} \frac{1}{3^n} = 2^{\frac{n}{2}} \frac{1}{3^n} \sum_{k=0}^n \frac{n!}{k!(n-k)!}$$

$$P = 2^{\frac{n}{2}} \frac{1}{3^n} 2^n = \left(\frac{2\sqrt{2}}{3}\right)^n = \left(\frac{2\sqrt{2}}{3}\right)^{12} = 0.49327$$

11)

- i) A recombining binominal tree or binominal lattice is one in which the sizes of the up-steps and down-steps are assumed to be the same under all states and across all time intervals. i.e.,  $u_t(j) = u$  and  $d_t(j) = d$  for all times  $t$  and states  $j$ , with  $d < \exp(r) < u$ . It therefore follows that the risk neutral probability 'q' is also constant at all times and in all states eg.  $q_t(j) = q$ . The main advantage of a 'n' period recombining binominal tree is that it has only  $[n+1]$  possible states of time as opposed to  $2^n$  possible states in a similar non-recombining binominal tree. This greatly reduces the amount of computation time required when using a binominal tree model. The main disadvantage is that the recombining binominal tree implicitly assumes that the volatility and drift parameters of the underlying asset price are

constant over time, which assumption is contradicted by empirical evidence



ii)

a) The risk-neutral probabilities at the first and second steps are as follows:

$$q_1 = (\exp(0.0175) - 0.95) / (1.10 - 0.95) = (0.06765) / 0.15 = 0.4510$$

$$q_2 = (\exp(0.025) - 0.90) / (1.20 - 0.90) = 0.41772$$

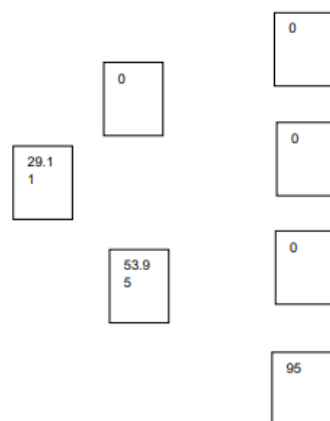
Put payoffs at the expiration date at each of the four possible states of expiry are 0,0,0 and 95.

Working backwards, the value of the option V1 (1) following an up step over the first 3 months is

$$V1(1) \exp(0.025) = [0.41772 * 0] + [0.58228 * 0] \text{ i.e., } V1(1) = 0$$

The value of the option V1 (2) following a down step over the first 3 months is:  $V1(2) \exp(0.025) = [0.41772 * 0] + [0.58228 * 95] \text{ i.e., } V1(2) = 53.9508$

The current value of the put option is:  $V0 \exp(0.0175) = [0.4510 * 0] + [0.5490 * 53.9508] \text{ i.e., } V0 = 29.105$



b) While the proposed modification would produce a more accurate valuation, there would be a lot more parameter values to specify. Appropriate values of  $u$  and  $d$  would be required for each branch of the tree and values of ' $r$ ' for each month would be required. The new tree would have  $2^6 = 64$  nodes in the expiry column. This would render the calculations prohibitive to do normally, and would require more programming and calculation time on the computer. An alternative model that might be more efficient numerically would be a 6-step recombining tree which would have only 7 nodes in the final column.

12) Given  $Z(t)$  is standard Brownian

a)  $dU(t) = 2dZ(t) - 0 = 0dt + 2dZ(t)$ .

Thus, the stochastic process  $\{U(t)\}$  has zero drift.

b)  $dV(t) = d[Z(t)]^2 - dt$ .  $d[Z(t)]^2 = 2Z(t)dZ(t) + \frac{2}{2} [dZ(t)]^2 = 2Z(t)dZ(t) + dt$  by the multiplication rule

Thus,  $dV(t) = 2Z(t)dZ(t)$ . The stochastic process  $\{V(t)\}$  has zero drift.

c)  $dW(t) = d[t^2 Z(t)] - 2t Z(t)dt$  Because  $d[t^2 Z(t)] = t^2 dZ(t) + 2tZ(t)dt$ , we have  $dW(t) = t^2 dZ(t)$ .

Thus, the process  $\{W(t)\}$  has zero drift.