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PROBABTILITY & STATISTICS - 1
ASSIGNMENT 2.

Q.1. Let x_i be claim size on home insurance and y_i be claim size on car insurance.

So, as given $\rightarrow X \sim N(800, 100^2)$
$Y \sim N(1200, 300^2)$

$$P(Y_1 + Y_2 + Y_3 > X_1 + X_2 + X_3 + X_4 + 800)$$

$$P(Y_1 + Y_2 + Y_3 - (X_1 + X_2 + X_3 + X_4) > 800)$$

$$(Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) \sim$$

$$N(3 \times 1200 - 4 \times 800), 300^2(3) + 100^2)$$

$$\sim N(400, 310000)$$

$$P\left(Z > \frac{800 - 400}{\sqrt{310000}}\right)$$

$$P(Z > 0.71842) \Rightarrow 1 - P(Z < 0.7184)$$

$$1 - P(Z < 0.72) = 0.23576$$

$\therefore$ Probability that after the next 4 home claims and 3 car claims the total size of car claims exceeds the total size of the home claims is 0.23576

Q.2.

$$P(N=n) = \binom{n+2}{n} 0.9^{3} 0.1^{n}$$

$N \sim$ Negative Binomial (k, p)

$$P(N=n) = {}^{n+2}C_n (0.9)^3 (0.1)^n$$

$$= {}^{(n+3)-1}C_{n-1} (0.9)^3 (0.1)^n$$

Thus, $k=3$, $p=0.9$

$$M_y t = M_N(\log M_x t)$$

$X \sim$ Gamma $(6, 2)$

$Y \to$ Total Claim Amount

i) MGF of Y.

$$M_y(t) = E(E k^{yt} / N=n)$$

$$E\left(\frac{e^{yt}}{N=n}\right) = E\left(e^{x_1 t + x_2 t + \dots x_n t}\right)$$

$$= E(e^{tx_1}) \cdot E(e^{tx_2}) \dots E(e^{tx_n})$$

$$= \left(1 - \frac{t}{2}\right)^{-6n}$$

$$\therefore E\left(E\left(\frac{e^{yt}}{N=n}\right)\right) = E\left(\left(1-\frac{t}{2}\right)^{-6n}\right)$$

$$= E\left(e^{N \log (1-t/2)^{-6}}\right).$$

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$$= M_N\left(\log\left(1-\frac{t}{2}\right)^{-6}\right)$$

$$M_N(t) = \left(\frac{pe^t}{1-qe^t}\right)^k = \left(\frac{pe^{\log\left(1-\frac{t}{2}\right)^{-6}}}{1-qe^{\log(1-t/2)^{-6}}}\right)^k$$

$$\left[\frac{p\left(1-t/2\right)^{-6}}{\cancel{t}-qe^{(1-t/2)^{-6}}}\right]^3 = \left[\frac{(0.9)\left(1-t/2\right)^{-6}}{1-0.1\left(1-t/2\right)^{-6}}\right]^3$$

ii) $V(Y) = E(N)\ V(X) + [E(X)]^2\ V(N)$

$$= \left[\frac{3}{0.9} \times \frac{6}{4}\right] + \left(\frac{36}{\cancel{x}}\right)^2 \times \frac{(3)(1-0.1)}{(0.9)^2}$$

$\therefore$ Standard Deviation $= 2.887$

Q.3. $f(x) = \lambda e^{-\lambda(x-\alpha)}$

$$MGF = E(e^{tx}) = \int_{\alpha}^{\infty} e^{tx}\lambda e^{-\lambda(x-\alpha)}\,dx$$

$$M_x t = \lambda\int_{\alpha}^{\infty} e^{tx} e^{-\lambda x} e^{\lambda\alpha}\,dx$$

$$= e^{\lambda\alpha}\lambda\int_{\alpha}^{\infty} e^{-(+\lambda-t)x}\,dx$$

$$\frac{e^{\alpha\lambda}\lambda}{t-\lambda}\left[e^{-(\lambda-t)x}\right]_{\alpha}^{\infty}$$

$$= \frac{e^{\lambda\alpha}\lambda}{t-\lambda}\left[-e^{-(\lambda-t)\alpha}\right]$$

$$\frac{e^{\lambda\alpha} \lambda e^{-\lambda\alpha} e^{t\alpha}}{\lambda - t}$$

$$= \boxed{e^{t\alpha} \frac{\lambda}{\lambda - t}}$$

(ii)

$$M_x(t) = \frac{\lambda e^{t\alpha}}{\lambda - t}$$

$$= \lambda e^{t\alpha} (\lambda - t)^{-1}$$

$$M_x'(t) = \lambda [e^{t\alpha}(\lambda - t)^{-2}(1) + (\lambda - t)^{-1} e^{t\alpha}(\alpha)]$$

$$= \lambda [(\lambda - t)^{-1} e^{t\alpha}(\alpha) + e^{t\alpha}(\lambda - t)^{-2}]$$

$$= \lambda e^{t\alpha} (\lambda - t)^{-2} [\alpha(\lambda - t) + 1]$$

Put $t = 0$

$$= \lambda(\lambda)^{-2} [\alpha(\lambda) + 1]$$

$$= \frac{1}{\lambda}[\lambda\alpha + 1]$$

$$= \frac{\lambda\alpha + 1}{\lambda}$$

$$M_x''(t) = \lambda [\alpha e^{t\alpha}(\lambda - t)^{-2} + \alpha^2 (\lambda - t)^{-1} e^{t\alpha} + e^{t\alpha}(-2)(\lambda - t)^{-3} + (\lambda - t)^{-2} e^{t\alpha}]$$

Put $t = 0$

$$\lambda [\alpha(\lambda)^{-2} + \alpha^2(\lambda)^{-1} + 2(\lambda)^{-3} + \alpha(\lambda)^{-2}]$$

$$\lambda \left[\frac{2\alpha}{\lambda^2} + \frac{\alpha^2}{\lambda} + \frac{2}{\lambda^3}\right]$$

$$E(x^2) = \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2}$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= \cancel{2\alpha}\ \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left[\alpha + \frac{1}{\lambda}\right]^2.$$

$$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda} - \left[\alpha^2 + \frac{1}{\lambda^2} + \frac{2\alpha}{\lambda}\right].$$

$$\cancel{\frac{2\alpha}{\lambda}} + \cancel{\alpha^2} + \frac{2}{\lambda^2} - \cancel{\alpha^2} - \frac{1}{\lambda^2} - \cancel{\frac{2\alpha}{\lambda}}.$$

$$\boxed{\therefore V(x) = \frac{1}{\lambda^2}}.$$

Q.4. $G_V(t) = \left(\frac{p}{1-qt}\right)^k$

$$G_V(t) = \sum t^v \times P(V = v).$$

$$P(V=4) = \frac{\Gamma_{k+4}}{\Gamma_5\,\Gamma_k} p^k q^k$$

$$= \frac{\Gamma_{k+4}}{4!\,\Gamma_k} p^k q^k$$

$$\boxed{P(V=4) = \frac{(k+3)!}{(k-1)!\,4!} p^k (1-q)^k.}$$

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Q.5 $f(x,y) = ax(x+y)$

(i) $P\left(y > \frac{1}{3}x + 10\right) =$

$$= a\int_{10}^{20}\int_{0}^{5} (x^2 + xy)\,dx\,dy = 1.$$

$$= \int_{10}^{20} \left(\frac{x^3}{3} + \frac{x^2}{2}y\right)_0^5 dy = \frac{1}{a}$$

$$= \int_{10}^{20} \left(\frac{125}{3} + \frac{25}{2}y\right) dy = \frac{1}{a}$$

$$= \left(\frac{125}{3}y + \frac{25}{4}y^2\right)_{10}^{20} = \frac{1}{a}$$

(ii) $$a\left(\frac{2500}{3} + 2500 - \frac{1250}{3} - 625\right) = 1.$$

$$a\left(\frac{1250}{3} + 1875\right) = 1.$$

$$a\left(\frac{6875}{3}\right) = 1.$$

$$a = \frac{3}{6875}$$

$$\boxed{a = 0.00043636}$$

(iii). $E(Y/X)$

$$f_{Y/X} = \frac{f(x,y)}{f(x)}$$

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$$f_x = \frac{3}{6875}\int_{10}^{20} (x^2 + xy)\,dy$$

$$= \frac{3}{6875}\left[x^2 y + \frac{xy^2}{2}\right]_{10}^{20}$$

$$= \frac{3}{6875}\left[20x^2 + x(200) - 10x^2 - 50x\right.$$

$$= \frac{3}{6875}\left[10x^2 + 150x\right]$$

$$= \frac{30x}{6875}\left[x + 15\right]$$

$$f_{y/x} = \frac{\cancel{3}}{\cancel{6875}} \times \frac{\cancel{x}(x+y)}{\cancel{30x}\ 10} \times \cancel{6875}(x+15)$$

$$= \frac{(x+y)(x+15)}{10}$$

$$E_{y/x} = \int_{10}^{20} \frac{y\cdot(x+y)}{10(x+15)}\,dy$$

$$= \frac{1}{10(x+15)}\int_{10}^{20} (xy + y^2)\,dy$$

$$= \frac{1}{10(x+15)}\left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_{10}^{20}$$

$$= \frac{1}{10(x+15)}\left[200x + \frac{8000}{3} - 50x - \frac{1000}{3}\right].$$

$$\frac{1}{10(x+15)}\left[150x+\frac{1000}{3}\right]$$

$$\boxed{\therefore E_{Y/X}=\frac{1}{x+15}\left[15x+\frac{100}{3}\right]}$$

Q.6. $X \sim Poi(\lambda)$ $\quad$ $Y \sim Poi(\mu)$

$M_X(t)=e^{\lambda(e^t-1)}$ $\quad$ $M_Y(t)=e^{\mu(e^t-1)}$

So, $Z=X-Y$

$$M_Z(t)=E(e^{tZ})=E(e^{t(X-Y)})=E(e^{tX}\cdot e^{-tY})$$

$$=E(e^{tX})\times E(e^{-tY})$$

$$=\frac{E(e^{tX})}{E(e^{tY})}$$

$$=\frac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}}$$

$$=e^{(\lambda-\mu)(e^t-1)}$$

$\therefore$ By Uniqueness MGF of $Z=e^{(\lambda-\mu)(e^t-1)}$

$\sim Poi(\lambda-\mu)$

Now, Let $Z=X+Y$

$$E(e^{tZ})=e^{\lambda(e^t-1)}\times e^{\mu(e^t-1)}$$

$$=e^{(\lambda+\mu)(e^t-1)}$$

By Uniqueness $\boxed{Z \sim Poi(\lambda+\mu)}$

Q.7.

			X		
		1	2	3	4
Y	1	0.2	0	0.05	0.15
	2	0	0.3	0.1	0.2

(i) $V(X/Y=2) = E(X^2/Y=2) - [E(X/Y=2)]^2$

$$E[X^2/Y=2] = \sum_{\text{all } x} x^2 \frac{P(X=x, Y=2)}{P(Y=2)}$$

when $x=1$;
$$\frac{(1)^2 P(X=1, Y=2)}{P(Y=2)}$$

When $x=2$
$$\frac{(2)^2 P(X=2, Y=2)}{P(Y=2)}$$

when $x=3$
$$\frac{(3)^2 P(X=3, Y=2)}{P(Y=2)}$$

when $x=4$
$$\frac{(4)^2 P(X=4, Y=2)}{P(Y=2)}$$

Adding all,
$$= \frac{(1)(0)}{0.6} + \frac{(4)(0.3)}{0.6} + \frac{(9)(0.1)}{0.6} + \frac{16(0.2)}{0.6}$$

$$= 8.8333$$

$\rightarrow$ $E(X/Y=2) = \sum_x x \frac{P(X=x, Y=y)}{P(Y=2)}$

$$\frac{(1)(0)}{0.6} + \frac{(2)(0.3)}{0.6} + \frac{(3)(0.1)}{0.6} + \frac{4(0.2)}{0.6}$$

$$= \frac{0.6+0.3+0.8}{0.6} = \underline{\underline{2.833}}$$

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(ii) $f(u,v) = \frac{48}{67}(2uv - u^2) \quad \ldots \quad (0 < u < 1;\ u < v < 2.)$

$E(U/V=v) = ?$

$\frac{f(u,v)}{f(v)} = f\ U/V=v.$

$f(v) = \frac{48}{67}\int_0^1 (2uv - u^2)\,du$

$= \frac{48}{67}\left[u^2 v - \frac{u^3}{3}\right]_0^1$

$= \frac{48}{67}\left[v - \frac{1}{3}\right]$

$= \frac{16}{67}[3v-1]$

$f(v) = \frac{16}{67}(3v-1)$

$\frac{f(u,v)}{f(v)} = \frac{\cancel{48}^{3}}{\cancel{67}}\frac{(2uv^2 - u^2)}{\cancel{16}(3v-1)} \cancel{\times 67}$

$= \frac{3}{3v-1}(2uv^2 - u^2) \cdot 2u^2v^2 - u^3$

$E(U/V=v) = \frac{3}{3v-1}\int_0^1 u(2uv^2 - u^2)\,du$

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$$\frac{3}{3v-1}\left[\frac{2u^2v^2}{3}-\frac{u^4}{4}\right]_0^1$$

$$=\frac{3}{3v-1}\left[\frac{2v^2}{3}-\frac{1}{4}\right]$$

$$=\frac{\cancel{3}}{3v-1}\left[\frac{8v^2-3}{\cancel{12}_4}\right]$$

$$=\frac{8v^2-3}{4(3v-1)}$$

$$\boxed{\therefore E(U/V=v)=\frac{8v^2-3}{4(3v-1)}}$$

Q.8. $E(X)=\frac{3}{2}$ $\quad V(X)=\frac{3}{4}$

$X\sim \text{Gamma}(3,2)$

$E(Y/X=x)=3x+1$ $\quad Var(Y/X=x)=2x^2+5$

$$E(Y)=E\left[E(Y/X=x)\right]$$
$$=E(3x+1)$$
$$=3E(X)+1$$
$$=3\times\frac{3}{2}+1$$

$$\boxed{E(Y)=\frac{11}{2}}$$

$$V(Y)=E\left[V(X/Y)\right]+V\left[E(X/Y)\right]$$

$$=E(2x^2+5)+V(3x+1)$$

$$=2E(X^2)+5+9V(X)$$

$$V(x) = E(x)^2 - [E(x)]^2$$

$$E(x) = \frac{3}{4} + \frac{9}{4}$$

$$E(x^2) = \frac{12}{4}$$

$$\boxed{E(x^2) = 3}$$

$$V(y) = 2(3) + 5 + 9(3/4)$$

$$= 11 + \frac{27}{4}$$

$$= \frac{44 + 27}{4}$$

$$\boxed{V(y) = \frac{71}{4}}$$

Q.9. $E(x) = 3$, $V(x) = 4$

$E(y) = 4$, $V(y) = 1$

Correlation coefficient $= -0.3 \sqrt{y}$

$= Cov(x, y)$

$= 0.6$

$$z = x + y$$

(i) $Cov(x, x+y) = Cov(x, x) + Cov(x, y)$

$= V(x) + Cov(x, y)$

$= 4 + (-0.6)$

$= 3.4$

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(ii) $Var(Z) = Cov(X+Y,\ X+Y)$

$= Cov(X,X) + 2Cov(X,Y) + Cov(Y,Y)$

$= V(X) + 2cov(X,Y) + V(Y)$

$= 4 + 2(-0.6) + 1.$

$= 5 - 1.2$

$= 3.8$

$\boxed{Var(Z) = 3.8}$

Q.10. $f(x,y) = k x^{-\alpha} e^{-y/\beta}$ $\quad 1 < x < \infty,\ 1 < y < \infty$

$k \int_1^{\infty} \int_1^{\infty} x^{-\alpha} e^{-y/\beta}\, dx\, dy = 1$ $\quad ,\ \alpha > 1,\ \beta > 0$

$k \int e^{-y/\beta} \left[\frac{x^{-(\alpha-1)}}{-\alpha+1} \right]_1^{\infty} = 1.$

$\frac{k}{1-\alpha} \int e^{-y/\beta} [-1]$

$\frac{k}{\alpha-1} \int e^{-y/\beta}\, dy = 1$

$\frac{k}{\alpha-1} \frac{\left[e^{-y/\beta} \right]_1^{\infty}}{(-1/\beta)}$

$\frac{-k}{\alpha-1} (\beta)(e^{-1/\beta}) = 1.$

$\boxed{k = \frac{\alpha-1}{(\beta)(e^{-1/\beta})}}$