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PROBABILITY & STATISTICS - ASSIGNMENT 2.

(1) Let X_i be claim size on home insurance.
Let Y_i be claim size on car insurance.

$$X \sim N(800, 100^2)$$
$$Y \sim N(1200, 300^2)$$

$P(Y_1 + Y_2 + Y_3 > X_1 + X_2 + X_3 + X_4 + 800)$

$P[Y_1 + Y_2 + Y_3 - (X_1 + X_2 + X_3 + X_4) > 800]$

$(Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) \sim N(3 \times 1200 - 4 \times 800;\ 300^2(3) + 100^2 \times 4) \sim N(400, 310000)$

$$P\left(Z > \frac{800 - 400}{\sqrt{310000}}\right)$$

$P(Z > 0.71842) = 1 - P(Z < 0.71842)$
$= 1 - P(Z < 0.72)$
$= 1 - 0.76424$
$= 0.23576.$

(2) $N \sim$ Negative Binomial (K, p)

$P(N = n) = {}^{n+2}C_n (0.9)^3 (0.1)^n$

so, $K = 3$ and $p = 0.9$

$M_y t = M_N(\log M_x t)$

$X \sim$ Gamma $(6, 2)$

$Y \rightarrow$ Total claim amount.

(a) MGF of Y.

$M_y(t) = E[E e^{yt} | N = n)$

$E[e^{yt} | N = n] = E(e^{X_1 t + X_2 t + \dots + X_n t})$
$= E(e^{tX_1}) \cdot E(e^{tX_2}) \dots E(e^{tX_n})$
$= \left(1 - \frac{t}{2}\right)^{-6n}$

so, $E(E(e^{ty} | N = n)] = E\left[\left(1 - \frac{t}{2}\right)^{-6n}\right]$
$= E\left[e^{N \log (1 - t/2)^{-6}}\right]$
$= M_N\left[\log (1 - t/2)^{-6}\right]$

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$$M_N(t) = \left(\frac{pe^t}{1-qe^t}\right)^k = \left[\frac{pe^{\log(1-t/2)^{-6}}}{1-qe^{\log(1-t/2)^{-6}}}\right]^k$$

$$= \left[\frac{(0.9)\left(1-\frac{t}{2}\right)^{-6}}{6-0.1\left(1-\frac{t}{2}\right)^{-6}}\right]^3$$

(ii) $V(Y) = E(N)V(X) + [E(X)]^2 \cdot V(N)$

$$= \left[\frac{3}{0.9} \times \frac{6}{4}\right] + \left(\frac{6}{2}\right)^2 \times \frac{(3)(0.1)}{(0.9)^2}$$

Standard Deviation $= 2.887$

3(i) $f(x) = \lambda e^{-\lambda(x-\alpha)}$

MGF $= E(e^{tx})$

$$= \int_\alpha^\infty e^{tx} \lambda e^{-\lambda(x-\alpha)}\, dx$$

$$M_x t = \lambda \int_\alpha^\infty e^{tx} e^{-\lambda x} e^{\lambda\alpha}\, dx$$

$$= e^{\lambda\alpha}\lambda \int_\alpha^\infty e^{-(\lambda-t)x}\, dx$$

$$= \frac{e^{\alpha\lambda}}{t-\lambda}\left[e^{-(\lambda-t)x}\right]_\alpha^\infty$$

$$= \frac{e^{\lambda\alpha}\lambda}{t-\lambda}\left[-e^{-(\lambda-t)\alpha}\right]$$

$$= \frac{e^{\lambda\alpha}\lambda}{\lambda-t} e^{-\lambda\alpha} e^{t\alpha}$$

$$= e^{t\alpha} \frac{\lambda}{\lambda-t}$$

(ii) $M_x(t) = \frac{\lambda e^{t\alpha}}{\lambda-t} = \lambda e^{t\alpha}(\lambda-t)^{-1}$

$$M_{x^2}(t) = \lambda\left[e^{t\alpha}(\lambda-t)^{-2} + (\lambda-t)^{-1} e^{t\alpha}(\alpha)\right]$$

$$= \lambda\left[(\lambda-t)^{-1} e^{t\alpha}(\alpha) + e^{t\alpha}(\lambda-t)^{-2}\right]$$

$$= \lambda e^{t\alpha}(\lambda-t)^{-2}\left[\alpha(\lambda-t)+1\right]$$

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$\boxed{t=0}$ $= \lambda(\lambda)^{-2}[\alpha(\lambda)+1]$

$= \frac{1}{\lambda}[\lambda\alpha+1]$

$= \frac{\lambda\alpha+1}{\lambda}$

$M_x''(t) = \lambda\left[\alpha e^{t\alpha}(\lambda-t)^{-2} + \alpha^2(\lambda-t)^{-1}e^{t\alpha} + e^{tx}(t\,2)(\lambda-t)^{-3} + (\lambda-t)^{-2}e^{t\alpha}\right]$

Put $t=0$

$\lambda\left[\alpha(\lambda)^{-2} + \alpha^2(\lambda)^{-1} + (2)(\lambda)^{-3} + \alpha(\lambda)^{-2}\right]$

$\lambda\left[\frac{2\alpha}{\lambda^2} + \frac{\alpha^2}{\lambda} + \frac{2}{\lambda^3}\right]$

$E(x^2) = \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2}$

$v(x) = E(x^2) - [E(x)]^2$

$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left[\alpha+\frac{1}{\lambda}\right]^2$

$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left(\alpha^2 + \frac{1}{\lambda^2} + \frac{2\alpha}{\lambda}\right)$

$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \alpha^2 - \frac{1}{\lambda^2} - \frac{2\alpha}{\lambda}$

$= \frac{1}{\lambda^2}$

4 $G_v(t) = \left(\frac{p}{1-qt}\right)^k$

$G_v(t) = \Sigma t^v\, P(V=v)$

$P(V=4) = \frac{\overline{|k+4}}{\overline{|5}\ \overline{|k}}\, p^k q^k$

$= \frac{\overline{|k+4}}{4!\ \overline{|k}}\, p^k q^k$

$= \frac{(k+3)!}{(k-1)!\,4!}\, p^k (1-p)^k$

5) $f(x,y) = ax(x+y)$

(i) $P\left(y > \frac{1}{3}x + 10\right)$

$$a\int_{10}^{20}\int_{0}^{5}(x^2+xy)\,dx\,dy = 1$$

$$\int_{10}^{20}\left[\frac{x^3}{3} + \frac{x^2}{2}y\right]_0^5 dy = \frac{1}{a}$$

$$\int_{10}^{20}\left(\frac{125}{3} + \frac{25}{2}y\right)dy = \frac{1}{a}$$

$$\left[\frac{125}{3}y + \frac{25}{4}y^2\right]_{10}^{20} = \frac{1}{a}$$

$$833.33 + 2500 - (416.667 + 625) = \frac{1}{a}$$

$$a = 0.00043636.$$

(ii) $P\left(y > \frac{1}{3}x + 10\right)$

$$f(y) = \frac{3}{6875}\int_0^5 x^2 + xy\,dx$$

$$= \frac{3}{6875}\left[\frac{x^3}{3} + \frac{x^2}{2}y\right]_0^5$$

$$= \frac{3}{6875}\left[\frac{125}{3} + \frac{25y}{2}\right]$$

so, $$\frac{3}{6875}\int_{\frac{1}{3}x+10}^{20}\left(\frac{125}{3} + \frac{25y}{2}\right)dy$$

$$= \frac{3}{6875}\left[\frac{125y}{3} + \frac{25y^2}{4}\right]_{\frac{1}{3}x+10}^{20}$$

$$= \frac{3}{6875}\left[\frac{125}{3}(20) + \frac{25}{4}(400) - \frac{125}{3}\left(\frac{1}{3}x + 10\right) - \frac{25}{4}\left(\frac{1}{3}x + 10\right)^2\right]$$

$$= \frac{3}{6875}\left[\frac{2500}{3} + 2500 - \frac{125x}{9} - \frac{1250}{3} - \frac{25}{4}\left[\frac{1}{9}x^2 + 100 + \frac{20x}{3}\right]\right]$$

$$= \frac{3}{6875}\left[\frac{2500}{3} + 2500 - \frac{125x}{9} - \frac{1250}{3} - \frac{25x^2}{36} - 625 - \frac{25x}{6}\right]$$

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(iii) $E(Y|X)$

$$f_{y|x} = \frac{f(x,y)}{f(x)}$$

$$f(x) = \frac{3}{6875}\int_{10}^{20} x^2+xy\,dy$$

$$= \frac{3}{6875}\left[x^2y + \frac{xy^2}{2}\right]_{10}^{20}$$

$$= \frac{3}{6875}\left[20x^2 + x(200) - 10x^2 - 50x\right]$$

$$= \frac{3}{6875}\left[10x^2 + 150x\right]$$

$$= \frac{30x}{6875}\left[x+15\right]$$

$$f_{y|x} = \frac{3}{6875}\,\frac{x(x+y)}{30x} \times 6875(x+15)$$

so, $$E(Y/X) = \int_{10}^{20} y\cdot\frac{(x+y)}{10(x+15)}\,dy$$

$$= \frac{1}{10(x+15)}\int_{10}^{20}(xy+y^2)\,dy$$

$$= \frac{1}{10(x+15)}\left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_{10}^{20}$$

$$= \frac{1}{10(x+15)}\left[200x + \frac{8000}{3} - 50x - \frac{1000}{3}\right]$$

$$= \frac{1}{10(x+15)}\left[150x + \frac{7000}{3}\right]$$

$$= \frac{1}{x+15}\left[15x + \frac{700}{3}\right]$$

(6) $X \sim Poi(\lambda)$ $\qquad$ $Y \sim Poi(\alpha)$

$M_x(t) = e^{\lambda(e^t-1)}$ $\qquad$ $M_y(t) = e^{\alpha(e^t-1)}$

so, $Z = X - Y$

$$M_z(t) = E(e^{tz}) = E(e^{t(X-Y)})$$

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$= E(e^{tx} \cdot e^{-ty})$

$= E(e^{tx}) \times E(e^{-ty})$

$= \dfrac{E(e^{tx})}{E(e^{ty})}$

$= \dfrac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}}$

$= e^{(\lambda-\mu)(e^t-1)}$

MGF of $Z = e^{(\lambda-\mu)(e^t-1)} \sim Pois(\lambda-\mu)$

Let $Z = X+Y$

$E(e^{tz}) = e^{\lambda(e^t-1)} \times e^{\mu(e^t-1)}$

$= e^{(\lambda+\mu)(e^t-1)}$

$Z \sim Pois(\lambda+\mu)$

(7)

		X			
		1	2	3	4
y	1	0.2	0	0.05	0.15
	2	0	0.3	0.1	0.2

(i) $V(X/Y=Z) = E(X^2/Y=Z) - [E(X/Y=Z)]^2$

$E[X^2/Y=Z] = \sum x^2 \dfrac{P(X=x, y=2)}{P(y=2)}$

where $x=1$, $x=2$, $x=3$, $x=4$

$\dfrac{(1)^2 P(X=1, y=2)}{P(y=2)} + \dfrac{(2)^2 P(X=2, Y=2)}{P(y=2)} + \dfrac{(3)^2 P(X=3, Y=2)}{P(y=2)} + \dfrac{(4)^2 P(X=4, Y=2)}{P(y=2)}$

$\dfrac{(1)(0)}{0.6} + \dfrac{(4)(0.3)}{0.6} + \dfrac{(9)(0.1)}{0.6} + \dfrac{(16)(0.2)}{0.6}$

$= 8.8333$

$E[X/Y=Z] = \sum_x x \dfrac{P(X=x, y=2)}{P(y=2)}$

$\dfrac{(1)(0)}{0.6} + \dfrac{(2)(0.3)}{0.6} + \dfrac{(3)(0.1)}{0.6} + \dfrac{(4)(0.2)}{0.6}$

$$= \frac{0.6 + 0.3 + 0.8}{0.6}$$

$$= 2.8333$$

$$\therefore V(X/Y = Z) = 8.8333 - (2.8333)^2$$
$$= 0.807.$$

(ii) $$f(u,v) = \frac{48}{67}(2uv - u^2)$$

$$\frac{f(u,v)}{f(v)} = f\, u|v = v$$

$$f(v) = \frac{48}{67}\int_0^1 (2uv - v^2)\,du$$

$$= \frac{48}{67}\left[u^2 v - \frac{u^3}{3}\right]_0^1$$

$$= \frac{48}{67}\left[v - \frac{1}{3}\right]$$

$$= \frac{16}{67}\left[3v - 1\right]$$

$$f(v) = \frac{16}{37}\left[3v - 1\right]$$

$$\frac{f(u,v)}{f(v)} = \frac{48}{67}\,\frac{(2uv^2 - v^2)}{\times 16(3v-1)} \times 67$$

$$= \frac{3}{3v-1}(2uv^2 - u^2).$$

$$E(U/V = v) = \frac{3}{3v-1}\int_0^1 v(2uv^2 - v^2)\,du$$

$$= \frac{3}{3v-1}\left[\frac{2u^3v^2}{3} - \frac{u^4}{4}\right]_0^1$$

$$= \frac{3}{3v-1}\left[\frac{2v^2}{3} - \frac{1}{4}\right]$$

$$= \frac{3}{3v-1}\left[\frac{8v^2 - 3}{12}\right] = \frac{8v^2 - 3}{4(3v-1)}$$

(8) $X \sim \text{Gamma}(3, 2)$

$E(Y/X = x) = 3x + 1$ $\qquad$ $\text{Var}(Y/X = x) = 2x^2 + 5$

$E(Y) = E[E(Y/X = x)]$

$= E(3X + 1)$

$= 3E(X) + 1$

$= 3\left(\frac{3}{2}\right) + 1$

$= \frac{9}{2} + 1$

$= \frac{9+2}{2}$

$= \frac{11}{2}$

$V(Y) = E[V(X/Y)] + V[E(X/Y)]$

$= E(2x^2 + 5) + V(3X + 1)$

$= 2E(X^2) + 5 + 9V(X)$

$V(X) = E(X^2) - [E(X)]^2$

$\frac{3}{4} + \frac{9}{4} = \{E(X)\}^2$

$E(X^2) = \frac{12}{4}$

$E(X^2) = 3$

$V(Y) = 2(3) + 5 + 9\left(\frac{3}{4}\right)$

$= 11 + \frac{27}{4}$

$= \frac{44 + 27}{4}$

$= \frac{71}{4}$

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(9) $E(X) = 3$ $\qquad$ $V(X) = 4$

$E(Y) = 4$ $\qquad$ $V(Y) = 1$

correlation coeff $= -0.3\sqrt{4}$ $\qquad = cov(X, Y)$

$= -0.6.$

$Z = X + Y$

(i) $cov(X, X+Y) = cov(X, X) + cov(X, Y)$

$= V(X) + cov(X, Y)$

$= 4 + (-0.6)$

$= 3.4$

(ii) $Var(Z) = cov(X+Y, X+Y)$

$= cov(X, X) + 2cov(X, Y) + cov(Y, Y)$

$= V(X) + 2cov(X, Y) + V(Y)$

$= 4 + 2(-0.6) + 1$

$= 5 - 1.2$

$= 3.8$

(10) $f(x, y) = kx^{-\alpha} e^{-y/\beta}$

$$k \int_1^\infty \int_1^\infty x^{-\alpha} e^{-y/\beta} \, dx \, dy = 1$$

$$k \int e^{-y/\beta} \left[\frac{x^{-(\alpha-1)}}{-\alpha+1}\right]_1^\infty = 1$$

$$\frac{k}{1-\alpha} \int e^{-y/\beta} [-1]$$

$$\frac{k}{\alpha-1} \int e^{-y/\beta} \, dy = 1$$

$$\frac{k}{\alpha-1} \frac{\left[e^{-y/\beta}\right]_1^\infty}{(-1/\beta)} \qquad -\frac{k}{\alpha-1}(\beta)\left(e^{-1/\beta}\right) = 1$$

$$k = \frac{\alpha - 1}{(\beta)\left(e^{-1/\beta}\right)}$$