Assignment 2

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1) Given:

$S_0 = £65$ $\quad C_0 = ?$

$\sigma = 25\%$ p.a.

$r = 2\%$ p.a.

$K = £55$

$T = 6 \text{ months} = 0.5$

(i) $C_t = S_0 \Phi(d_1) - Ke^{-rT}\Phi(d_2)$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_1 = \frac{\ln\left(\frac{65}{55}\right) + \left(2\% + \frac{20\%^2}{2}\right)0.5}{20\%\sqrt{0.5}}$$

$\Rightarrow \boxed{d_1 = 1.08996}$

$\Rightarrow d_2 = d_1 - \sigma\sqrt{T}$

$\Rightarrow \boxed{d_2 = 0.91318}$

$\therefore C_t = 65\,\Phi(1.09) - 55e^{-0.02\times0.5}\,\Phi(0.9131)$

$C_t = 11.4141$

$$\boxed{C_t = 11.4646}$$

(ii) $\Delta = \dfrac{\text{Change in price of call option}}{\text{Change in price of underlying}}$

(iii)

$\Delta_c = \Phi(d_1)$

$\Delta_p = -\Phi(-d_1)$

$\Delta_c = 0.86214$

(iv) $\boxed{\Delta_p = -0.13786}$

2(i) $\Delta = \dfrac{\text{Change in price of derivative}}{\text{Change in price of underlying asset}}$

rate of change of option price with respect to change in price of underlying asset.

$\nu = \dfrac{\text{Change in price of option}}{\text{Change in volatility of underlying asset}}$.

rate of change of option price w.r.t. change in volatility of underlying asset.

(ii) Put call parity:

$$C - P = S - K e^{-rT}$$

$$\frac{d}{d\sigma}(C-P) = \frac{d}{d\sigma}\left(S - K e^{-rT}\right)$$

$$\Rightarrow V_c - V_p = 0$$

($\because$ S & K are constants)

$$\Rightarrow \boxed{V_c = V_p}$$

(iii) $S_0 = \$55$

$X = \$50$

$\sigma = 25\%$ p.a.

$r = 5\%$

$T = 1$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$= \frac{\ln\left(\frac{55}{50}\right) + \left(5\% + \frac{1}{2} \times 25\%^2\right)}{25\%}$$

$$\boxed{d_1 = 0.70624}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$= 0.70624 - 25\%$

$$\boxed{d_2 = 0.4562}$$

$$C_t = S_0 \Phi(d_1) - K e^{-rT} \Phi(d_2)$$

$$= 55(0.76115) - 50 e^{-0.05}(0.67724)$$

$$\boxed{C_t = \$9.65}$$

$$C - p = S - K e^{-rT}$$

$$\Rightarrow 9.65 - p = 55 - 50 e^{-0.05}$$

$$\boxed{P = \$2.211}$$

(iv) Delta hedging:

A portfolio for which the overall delta is equal to 0 is decribed as delta hedged or delta neutral. The portfolio is immune to changes in the price of the underlying asset.

Vega hedging:

→ A portfolio for which the overall vega is equal to 0 is decribed as vega hedged or vega neutral. The portfolio is immune to small changes in the assumed level of volatility.

(v) x units of call option
Y units of put option -
Z units of forward

	C.F. @ PV	Δ	V
Call	$c = 9.65$	Δ_c	V_c
Put	$p = 2.211$	Δ_p	V_p
forward	-	1	0

Δ of forward = 1
V of forward = 0

Vega Hedging:

$$x V_c + y V_p + z V_F = 0$$

$$(\because V_F = 0)$$

$$\Rightarrow x V_c + y V_p = 0$$

$$\because (V_c = V_p)$$

$$\Rightarrow V_c (x+y) = 0 \Rightarrow x + y = 0$$

$$\Rightarrow \boxed{y = -x} \quad (i)$$

Delta Hedging:

$$x \Delta_c + y \Delta_p + z \Delta_F = 0$$

$(\Delta_F = 1)$

$x\Delta_c + y\Delta_p + z = 0$

$(\because y = -x, \ \Delta_p = \Delta_c - 1)$

$\Rightarrow x\Delta_c - x(\Delta_c - 1) + z = 0$

$\Rightarrow x + z = 0$

$\Rightarrow \boxed{x = -z}$ (ii)

from (i) & (ii),

$x = -y = -z$

portfolio = 1000

$x.C + y.P + z.F = 1000$

$(\because F = 0)$

$\Rightarrow x.C + y.P = 1000$

$\Rightarrow x.9.6525 + y.2.2140 = 1000$

$\Rightarrow 9.6525x - 2.2140x = 1000$

$\Rightarrow 7.4385x = 1000$

$\Rightarrow \boxed{x = 134.43}$

$$\Rightarrow \boxed{x = 134.43}$$

$$\boxed{y = z = -134.43}$$

$\therefore$ portfolio -

1) Long position on 134 call options
2) short position " 134 put options
3) short position " 134 forward

Q3)

(i) $C_t = E\left(e^{-r(T-t)} C_T \middle| F_t\right)$ -

$F_t \rightarrow$ Filtration @ $t > 0$

$C_T \rightarrow$ payoff under derivative at T

$C_t \rightarrow$ derivative value at time t

(ii) $S_0 = £50$

$X = £49$

$r = 5\%$ p.a.

$\sigma = 25\%$ p.a.

$T = 0.5$

$$C_t = S_0 \Phi(d_1) - Ke^{-rT} \Phi(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$d_1 = \frac{\ln\left(\frac{50}{49}\right) + \left(5\% + 12.5\%\right)0.5}{25\%\sqrt{0.5}}$$

$$= \boxed{0.347}$$

$$d_2 = d_1 - 25\% \times \sqrt{0.5}$$

$$\boxed{d_2 = 0.1673}$$

$$\Phi(d_1) = 0.6346$$

$$\Phi(d_2) = 0.5664$$

$$C_T = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

$$= 50 \times 0.6346 - 49e^{-0.05 \times 30}(0.5664)$$

$$\boxed{C_T = 4.66}$$

(iii) Same as european call = 4.66

(iv) $S_t + p = C + Ke^{-rT}$

$p = 4.66 + 49 e^{-0.05 \times .25} - 50$

$\boxed{p = 2.95}$

(V) If the stock pays a dividend then the price of the underlying would fall.

This would lead to a fall in the price of the European call and a rise in price of the European put.

Due to the potential early exercise opportunity, the American call would be more expensive than the European call.

Q4 $Z_t \sim$ SBM under P

$\gamma_t \rightarrow$ previsible process

∴ There exists a measure Q equivalent to P where,

$$\tilde{Z}_t = Z_t + \int_0^t \gamma_s \, ds$$

$\tilde{Z} \rightarrow$ SBM ... Q

$\tilde{Z}_t \rightarrow$ SBM under $\mathbb{Q}$.

If Z_t is a SBM under P and if $\mathbb{Q}$ is equivalent to P then there exists a previsible process γ_t such that

$$\tilde{Z}_t = Z_t + \int_0^t \gamma_s \, ds$$

$\tilde{Z}_t \rightarrow$ SBM under $\mathbb{Q}$

(ii) The discounted value of asset prices are martingales.

Q5)

(i) $\boxed{\Delta = \Phi(d_1)}$

$$= \frac{\text{Change in price of underlying asset}}{\text{Change in price of call option}}$$

(ii) $$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$\Rightarrow 0.3 = \ln\left(90 \right) + \left(0.02 + 0.5 \times \sigma^2\right)5$

$$\Rightarrow 0.3 = \frac{\ln\left(\frac{90}{95.91}\right) + (0.02 + 0.5 \times \sigma^2)5}{\sigma\sqrt{5}}$$

$$\Rightarrow \sigma\, 0.3\sqrt{5} = -0.1378 + (0.1 + 2.5\sigma^2)$$

$$\Rightarrow \sigma\, -2.5\sigma^2 = -0.05634$$

$$\Rightarrow 2.5\sigma^2 - 0.67083\sigma - 0.0378 = 0$$

$$\Rightarrow \sigma = \frac{0.67083 \pm \sqrt{0.67083^2 - 4 \times 2.5 \times (0.0378)}}{2 \times 2.5}$$

$$\Rightarrow \boxed{\sigma = 32\%}$$

(iii) $R_0 = \$30$

$\sigma = \sqrt{15\%}\ p.a.$

Option payoff = c at time T if

$$\frac{S_1}{S_0} > k_S,\quad \frac{R_1}{R_0} < k_R$$

$$\text{payoff} = c\,e^{-rT}\, g\left(\frac{S_1}{S_0} < k_S\right) g\left(\frac{R_1}{R_0} < k_R\right)$$

(iv) perfect corr. $\rightarrow$

$$\left(\frac{R_1}{R_0} = \frac{S_1}{S_0}\right)$$

perice =

$$C e^{-rT} q\left(R_1 / R_0 < K_S\right) =$$

$$C e^{-rT} q\left(S_1 / S_0 < K_S\right)$$

$$\therefore \text{price} = C e^{-rT} \cdot q\left(\frac{R_1}{R_0} < \min(K_S, K_R)\right)$$

$$= c e^{-rT} q \cdot \left(\frac{S_1}{S_0} < \min(K_S, K_R)\right)$$

(v) price =

$$C e^{-rT} q\left(\frac{S_1}{S_0} < K_S\right) \cdot q\left(\frac{R_1}{R_0} < K_R\right)$$

$$= 50 e^{-0.02 \times 1} \cdot q(S_1 < 32) \cdot q(R_1 < 18)$$

$$= \$1.61$$

6 (i) a) $\Delta = -\Phi(-d_1)$

$= \Phi(d_1) - 1$

b) $\Delta = 0$

$100{,}000\,\Delta + 24830 = 0$

$\Delta = \dfrac{-24830}{100{,}000}$

$\boxed{\Delta = -0.2483}$

(ii) $\Phi(-d_1) = 0.2483$

$\Rightarrow 1 - \Phi(d_1) = 0.2483$

$\Rightarrow \Phi(d_1) = 0.7517$

$\Rightarrow d_1 = 0.68$

$$0.68 = \frac{\ln\left(\frac{0.64}{0.63}\right) + \left(0.03 + \frac{1}{2}\sigma^2\right)}{\sigma}$$

$\boxed{\sigma = 7.1\%}$

(iii) a) $Xe^{-rT}\Phi(-d_2) - S_0\left[\Phi(-d_1)\right] = 0.2483$

$= 0.63 \times e^{-0.03} \times [1 - \Phi(0.609)] - 0.64 \times 0.2483$

$= \boxed{\cancel{6.894}}$

b) Total cash holding

$= 24830 \times 0.64 + 100{,}000 \times 0.6894$

$= £165{,}806$

Q7(i) $\Delta = \frac{\partial f}{\partial S} = \frac{\partial f}{\partial S}(t, S_t)$

$f \rightarrow$ derivative
$S \rightarrow$ underlying asset

$\gamma = \frac{\partial^2 f}{\partial S^2}$

$\nu = \frac{\partial f}{\partial \sigma_S}$

(ii) $X = 50$

$c = 17.91$

$S_0 = 60$

$\sigma = 25\%$ p.a.

$T = 3$ years

$r = 3\%$ p.a.

$v = \$29$

$\boxed{\Delta = 0.801}$

(iii) 0.801 sharars

$c - \Delta \times S_0$ short cash $=$ 30.15 short in cash

(iv) $f(S, \sigma + \delta) = f(S, \sigma) + \delta \frac{df}{d\sigma}$

$= 17.91 + 29 \times 0.02$

$\boxed{= 18.49}$

Q8 $\Delta = \frac{\partial f}{\partial S}$

(ii) $S_0 = \$100$

$r = 3\%$

$X = 109.42$

$t = 1$

$\Delta = 0.42074$

$\rightarrow \Phi(d_1) = 0.42074$

$$\boxed{d_1 = -0.2}$$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$$

$$\Rightarrow -0.2 = \frac{\ln\left(\frac{100}{109.42}\right) + \left(0.03 + \frac{\sigma^2}{2}\right)}{\sigma}$$

$$\Rightarrow -0.2\sigma + 0.09 - 0.03 - \frac{\sigma^2}{2} = 0$$

$$\Rightarrow -\frac{\sigma^2}{2} - 0.2\sigma + 0.06 = 0$$

$$\boxed{\sigma = 0.2}$$

Q 9 (i) PDE for g :

$$\frac{\partial g}{\partial t} + (r-q)S_T\frac{\partial g}{\partial S_T} + \frac{1}{2}\sigma^2 S_t^2\frac{\partial^2 g}{\partial S_t^2} = rg$$

Boundary condn :

$$D_T = g(T, S_T) = f(S_T)$$

(ii) Price of derivative @ time t & value for μ

$$D_T = g(t, S_t) = \frac{S_t^n}{S_0^{n-1}} e^{\mu(T-t)} \quad (n > 1)$$

Then, $D_T = g(T, S_T)$

$$= f(S_T) = \frac{S_T^n}{S_0^{n-1}}$$

$$\frac{\partial g}{\partial t} = -\mu \frac{S_t^n}{S_0^{n-1}} e^{\mu(T-t)} = -\mu g$$

$$\frac{\partial g}{\partial S_t} = \frac{n S_t^{n-1}}{S_0^{n-1}} e^{\mu(T-t)} = \frac{n}{S_t} g$$

$$\frac{\partial^2 g}{\partial S_t^2} = \frac{n(n-1) S_t^{n-2}}{S_0^{n-1}} e^{\mu(T-t)} = \frac{n(n-1)}{S_t^2} g$$

$$-\mu g + (r-q) S_t \frac{n}{S_t} g + \frac{1}{2}\sigma^2 S^2 n(n-1)$$

$$-\mu g + (r-q) S_t \frac{n}{S_t} g + \frac{1}{2}\sigma^2 S_t^2 \frac{n(n-1)}{S_t^2} g = rg$$

$$-\mu + (r-q)n + \frac{1}{2}\sigma^2 n(n-1) = r$$

$$\mu = (r-q)n - r + \frac{1}{2}\sigma^2 n(n-1)$$

If $q = 0$

$$\mu = rn - r + \frac{1}{2}\sigma^2 n(n-1)$$

$$= r(n-1) + \frac{1}{2}\sigma^2 n(n-1)$$

$$= \left(r + \frac{1}{2}\sigma^2 n\right)(n-1)$$

9.10(1) Put Call parity for stockpaying no dividends:

$$c_t + Ke^{-r(T-t)} = S_t + p_t$$

(ii) $X = 120$

$T = 1$

$C = 10.09$

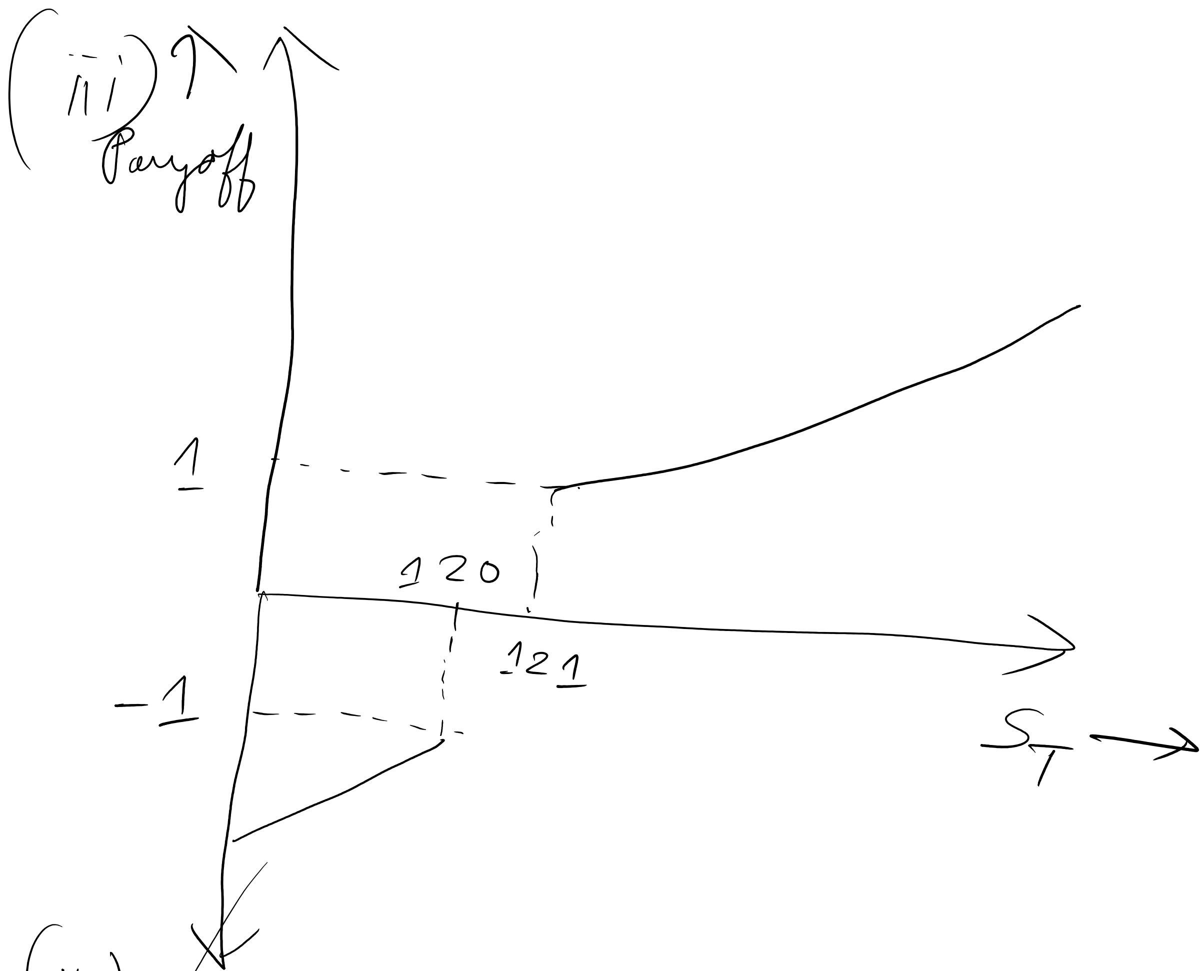
$r = 2\%$ p.a

$S_0 = 110$

$$C = S_0\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2)$$

$$\boxed{d_2 = d_1 - \sigma}$$

(iii)



(iv)

$$S_1 - 121 \le D \le S_1 - 120$$

$$\Rightarrow \quad S_0 - 121e^{-r} \leq V \leq S_0 - 120e^{-r}$$

Q11 $X = 150$

$r = 2\%$ p.a.

$S_0 = 117.98$

Q12 $S_0 = 8$

$X = 9$

$r = 2\%$ p.a

$\sigma = 20\%$ p.a.

$T = 3/12$