Aarushi Shinde 411

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Financial Engineering - 1

Unit 1 & 2

Q1.

(i) Arbitrage opportunity is a situation where we can make a certain profit with no risk. This is sometimes described as a free lunch.

An arbitrage opportunity means that:
(a) we can start at time 0 with a portfolio that has a net value of zero (implying that we are long in some assets and short in others.) This is usually called a zero-cost portfolio.

(b) at some future time T:
- the probability of a loss is 0.
- the probability that we make a strictly positive profit is greater than 0.

If such an opportunity existed then we could multiply up this portfolio as much as we wanted to make as large a ~~porfi~~ profit as we desired.

(ii) Law of one price.

The Law of one price states that any two portfolios that behave in exactly the same way must have the same price. If this were not true, we could buy the 'cheap' one and sell the 'expensive' one to make an arbitrage (risk-free) profit.

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(iii)

(a) Using Put - Call parity, the value of put option should be :-

$$P_1 = C_1 + K\exp[-r(T-t)] - S_t \exp[-q(T-t)]$$

$$= 30 + 120\exp(-0.5 \times 0.25)$$

$$= 28.11.$$

(b) Arbitrage profit

If the put options are only Rs. 23 then they are cheap. If things are cheap then we buy them.

So looking at the put-call parity relationship, we "buy the cheap side and sell the expensive side", i.e we buy put options and shares and sell call options and cash.

For example :-

- Sell 1 call option Rs. 30
- Buy 1 put option (Rs. 23)
- ~~Bay~~ Buy 1 share (Rs. 125)
- Sell (borrow) cash Rs. 118

This is a zero-cost portfolio and because put-call parity does not hold, we know it will make an arbitrage profit. We can check as follows :-

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In 3 months time repaying the cash will cost us:

$$118 \exp\left(\frac{0.05 \times 3}{12}\right) = Rs\ 119.48$$

We also receive dividends d on the share.

1. If the share price is above 120 in 3 months time then the party will exercise their call option and will have to give them the share. They will pay 120 for it and our profit is:-

$$120 - 119.48 + d = 0.52 + d$$

2. If the share price is below 120 in 3 months time then we will exercise our put option and sell it for 120. Our profit is:-

$$\cancel{120 + i} \quad 120 - 119.48 + d = 0.52 + d$$

(The call option is useless to the other party and will expire worthless).

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Q4.

Let a ex-dividend dates are anticipated for a stock and t the times before which the stock goes ex-dividend. Dividends are denoted by $d_1, \dots d_n$

If the option is exercised prior to the ex-dividend date then the investor receives $S(t_n) - K$.

If the option is not exercised, the price drops to $S(t_n) - d_n$.

The value of the american option is greater than $S(t_n) - d_n - K\exp[-r(T-t_n)]$

It is never optional to exercise the option if $S(t_n) - d_n - K\exp[-r(T-t_n)] >= S(t_n) - K$ i.e.

$$d_n <= k * (1 - \exp[-r(T-t_n)]$$

Using this equation:- We have $k \times (1 - \exp(-r(T-t_n))$

$= 350 \times [1 - e^{-0.95 \times (0.8333 - 0.25)}]$

$= 18.87$

and.

$65 \times [1 - \exp(-0.95) \times (0.8333 - 0.25)]$

$= 10.91.$

Hence, it is never optimal to exercise the american option on the two ex-dividend rates.

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Q3.
(i) Given :-

$$F(t, x) = e^{-t} x^2$$

$$\frac{df}{dt} = -e^{-t} x^2 = -f$$

$$\frac{df}{dx} = 2e^{-t} x .$$

$$\frac{df}{dx^2} = 2e^{-t}$$

$$Y_t = f(t, X_t)$$

Applying Ito's lemma.

$$dY_t = \frac{df}{dt} dt + \frac{df}{dx} dX_t + \frac{1}{2} \frac{d^2 f}{dx^2} \sigma^2 X_t^2 dt$$

$$= -f dt + 2e^{-t} X_t dX_t + e^{-t} \sigma^2 X_t^2 dt$$

$$= -Y_t dt + 2e^{-t} X_t^2 \frac{dX_t}{\cancel{X_t}} + e^{-t} \sigma^2 X_t^2 dt$$

$$= -Y_t dt + 2Y_t . (0.25 dt + \sigma d W_t) + \sigma^2 Y_t dt$$

$$\frac{dY_t}{Y_t} = [2 \times (0.25) - 1 + \sigma^2] dt + 2\sigma d W_t$$

$$= [\sigma^2 - 0.5] dt + 2\sigma d W_t$$

$$\therefore \; dY_t = [\sigma^2 - 0.5] Y_t dt + 2\sigma Y_t dW_t$$

(ii) The process is martingale if drift is zero. This means $\sigma^2 - 0.5 = 0$

i.e. $\sigma^2 = 0.5$

Q5.

The required probability is the probability of the stock price being greater than Rs. 258 in 6 months time.

The stock price follows Geometric brownian motion. i.e. $S_t = S_0 \exp(\mu - \sigma^2/2)t + \sigma W_t$

$\therefore \ln(S_t)$ follows normal distribution with mean $\ln(S_0) + (\mu - \sigma^2/2)t$ and variance $\sigma^2 t$.

Implies $\ln(S_t)$ follows $\phi\left[\ln 254 + \left(0.16 - \frac{0.35^2}{2}\right)(0.35 \times 0.5)^{1/2}\right]$

$= \phi(5.59, 0.247)$

This means $\left[\ln(S_t) - \frac{\mu(S_t)}{\sigma t^{(1/2)}}\right]$ follows standard normal distribution. Hence the probability that stock price will be higher than the strike price of Rs. 258 in 6 months time =

$1 - \frac{N(5.55 - 5.59)}{0.247}$

$= 1 - N(-0.1364)$

$= 0.5542.$

The put option is exercised if the stock price is less than Rs. 258 in 6 months time.

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The probability of this $= 1 - 0.5542$

$= 0.4457.$

Q6.

(i) The given relationship can be written as:

$$S_t = S_0 e^{\mu t + \sigma B_t}$$

Since S_t is a function of standard Brownian motion, B_t applying Ito's Lemma, the SDE for the underlying stochastic process becomes:

$$dB_t = 0 \times dt + 1 \times dB_t.$$

Let $G(t, B_t) = S_t = S_0 e^{\mu t + \sigma B_t}$, then

$$dG/dt = \mu S_0 e^{\mu t + \sigma B_t} = \mu S_t$$
$$dG/dB_t = \sigma S_0 e^{\mu t + \sigma B_t} = \sigma S_t$$
$$d^2G/dB^2_t = \sigma^2 S_0 e^{\mu t + \sigma B_t} = \sigma^2 S_t$$

Hence, using Ito's Lemma from tables:-

$$dG = \left[0 \times \sigma S_t + \frac{1}{2} \times 1^2 \times \sigma^2 S_t\right] dt + 1 \times \sigma S_t dB_t$$

$$dS_t = \left(\mu + \frac{1}{2}\sigma^2\right) S_t \, dt + \sigma S_t \, dB_t$$

Thus,

$$\frac{dS_t}{S_t} = \sigma \, dB_t + \left(\mu + \frac{1}{2}\sigma^2\right) dt$$

So,

$$c_1 = \sigma \quad \text{and} \quad c_2 = \mu + \frac{1}{2}\sigma^2$$

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(ii) The expected value of S_t is:

$$E[S_t] = E[S_0 e^{\mu t + \sigma B_t}]$$

$$= S_0 e^{\mu t} E[e^{\sigma B_t}]$$

$\because B_t \sim N(0,1)$ its MGF is $E[e^{\theta B_t}] = e^{1/2 \theta^2 t}$

So,

$$E[S_t] = S_0 e^{\mu t} \times e^{1/2 \sigma^2 t}$$

$$= S_0 e^{\mu t + 1/2 \sigma^2 t}.$$

The variance of S_t is:-

$$\begin{aligned}
Var[S_t] &= E[S^2_t] - E[S_t]^2 \\
&= E[S_0^2 e^{2\mu t + 2\sigma B_t}] - (S_0 e^{\mu t + 1/2 \sigma^2 t})^2 \\
&= S_0^2 e^{2\mu t} E[e^{2\sigma B_t}] - S_0^2 e^{2\mu t + \sigma^2 t} \\
&= S_0^2 e^{2\mu t + 2\sigma^2 t} - S_0^2 e^{2\mu t + \sigma^2 t} \\
&= S_0^2 e^{2\mu t} (e^{2\sigma^2 t} - e^{\sigma^2 t})
\end{aligned}$$

(iii) $Cov[S_{t_1}, S_{t_2}] = E[S_{t_1}, S_{t_2}] - E[S_{t_1}] E[S_{t_2}]$

From above,

$E[S_{t_1}] = S_0 e^{\mu t_1 + 1/2 \sigma^2 t_1}$ and

$E[S_{t_2}] = S_0 e^{\mu t_2 + 1/2 \sigma^2 t_2}$

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The expected value of the product is :-

$$E[S_{t1}, S_{t2}] = E\left[S_0 \exp(\mu t_1 + \sigma B_{t1})\, S_0 \exp(\mu t_2 + \sigma B_{t2})\right]$$

$$= S_0^2\, e^{\mu(t_1+t_2)}\, E\left[\exp(\sigma B_{t1} + \sigma B_{t2})\right]$$

To evaluate this we need to split B_{t2} into two independent components :-

$B_{t_2} = B_{t_1} + (B_{t_2} - B_{t_1})$ where $B_{t_2} - B_{t_1} \sim N(0, t_2 - t_1)$

Hence,

$$E[S_{t_1}, S_{t_2}]$$

$$= S_0^2\, e^{\mu(t_1+t_2)}\, E\left[\exp(\sigma B_{t_1} + \sigma\{B_{t_1} + (B_{t_2} - B_{t_1})\})\right]$$

$$= S_0^2\, e^{\mu(t_1+t_2)}\, E\left[\exp(2\sigma B_{t_1} + \sigma\{B_{t2} - B_{t1}\})\right]$$

$$= S_0^2\, e^{\mu(t_1+t_2)}\, E\left[\exp(2\sigma B_{t1})\right] + E\left[\exp\{B_{t_2} - B_{t_1}\}\right]$$

$$= S_0^2\, e^{\mu(t_1+t_2)}\, \exp(2\sigma^2 t_1) \exp\left[1/2\, \sigma^2 (t_2 - t_1)\right]$$

$$= S_0^2\, e^{\mu(t_1+t_2)}\, \exp\left(\frac{3}{2}\sigma^2 t_1 + \frac{1}{2}\sigma^2 t_2\right)$$

Putting all the equations together :-

$$Cov[S_{t1}, S_{t2}] = S_0^2\, e^{\mu(t_1+t_2)}\left[\exp(3/2\,\sigma^2 t_1) - \exp(1/2\,\sigma^2 \cdot\right.$$
$$\exp\left(\frac{1}{2}\sigma^2 t_2\right)$$

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Q7.

(i) Setting up the commodity tree using u for up and d for down move, p is up-step ~~prop~~ probability:-

$$S_0 \xrightarrow{\ p\ } S_0 u$$

$$S_0 \xrightarrow{\ 1-p\ } S_0 d$$

Where p is the up probability and (1-p) the down probability.

Then

$$E(C_t) = S_0\left[pu + (1-p)d\right]$$

$$Var(C_t) = E(C_t^2) - E(C_t)^2$$

$$= S_0^2\left[pu^2 + (1-p)d^2\right] - S_0^2\left[pu + (1-p)d\right]^2$$

$$= S_0^2\left[pu^2 + (1-p)d^2 - (pu + (1-p)d)^2\right]$$

$$= S_0^2\left[p(1-p)u^2 + p(1-p)d^2 - 2p(1-p)\right]$$

$$= S_0^2\, p(1-p)(u-d)^2$$

Equating moments:

$$S_0 e^{rt} = S_0\left[pu + (1-p)d\right] \quad \text{--- (A)}$$

And

$$\sigma^2 S_0^2 t = S_0^2\, p(1-p)(u-d)^2 \quad \text{--- (B)}$$

From (A) we get,

$$p = \frac{e^{rt} - d}{u - d} \quad \text{——} \quad (C)$$

Substituting p into equation B, we get

$$\sigma^2 t = \frac{e^{rt} - d}{u - d}\left(1 - \frac{e^{rt} - d}{u - d}\right)(u-d)^2$$
$$= -(e^{rt} - d)(e^{rt} - u)$$
$$= (u+d)\, e^{rt} - (1 + e^{2rt})$$

Putting $d = \frac{1}{u}$ and multiplying through by u we get,

$$u^2 e^{rt} - u\,(1 + e^{2rt} + \sigma^2 t) + e^{rt} = 0$$

This is a quadratic in u which can be solved in the usual way.

(ii)

(a) $\sigma = 0.15$, $t = 0.25 \Rightarrow u = \exp(0.15 \times \sqrt{0.25})$
$$= \exp(0.075)$$
$$= 1.077884$$

$$d = \frac{1}{u} = 0.92774$$

t = 0; 0.25; 0.5; 0.75

80 → 86.231, 74.22

86.231 → 92.947, 80.001

74.22 → 80.001, 68.857

92.947 → 100.186 Node A, 86.23 Node B

80.001 → 86.23 Node B, 74.22 Node C

68.857 → 74.22 Node C, 63.882 Node D

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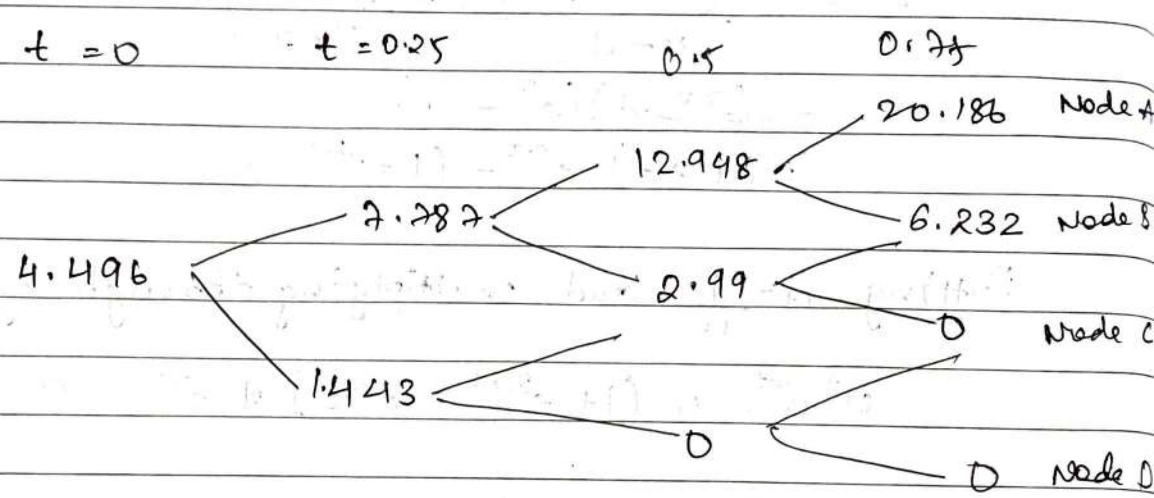
b) $r = 0$, we have

$$p = \frac{e^{rt} - d}{u - d} = \frac{1 - 0.927744}{(1.077884 - 0.927744)} = 0.48126$$

Discounting back the final payoff at $t = 0.75$ to $t = 0$ along the tree using p and $1-p$, we get



Hence value of the call option is 4.496.

c) The lookback calls pays the difference between the minimum value and the final value.

Notate paths by U for up and D for down, in order
We get the payoffs.

UUU	$(100.186 - 80) = 20.186$	A
UDU	$86.232 - 80 = 6.232$	B
UUD	$86.232 - 80 = 6.232$	B
UDD	$74.22 - 74.22 = 0$	C
DUU	$86.232 - 74.22 = 12.012$	B

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DUD	74.22 - 74.22	= 0	C
DDU	74.22 - 68.857	= 5.363	C
DDD	63.882 - 63.882	= 0	D

The lookback payoffs are, for each successful path (i.e. with a non-zero result).

Probabilities of arriving at each node are:

$A = p^3 = 0.11147$
$B = p^2(1-p) = 0.12015$
$C = p(1-p)^2 = 0.12950$
$D = p(1-p)^3 = 0.13959$

∴ The value of lookback option is :-

$(0.11147 \times 20.186) + [0.12015(6.232 + 6.232 + 12.012)] + (0.12950 \times 5.363)$

$= 5.8854$

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Q8.

(i) Consider a stock whose current price is S_0 and an option who current price is f. We suppose that the option lasts for time T and that during the life of the option the stock price can either move up from S_0 to a new level $S_0 u$ or move down to $S_0 d$ where $u>1$ and $d<1$.

Let the payoff be f_u if the stock price becomes $S_0 u$ and f_d if stock price becomes $S_0 d$.

Let us construct a portfolio which consists of a short position in the option and a long position in Δ shares. We calculate the value of Δ that makes the portfolio risk-free.

Now, if there is an upward movement in the stock, the value of the portfolio becomes $\Delta S_0 u - f_u$ and if there is a downward movement of stock, the value of the portfolio becomes $\Delta S_0 d - f_d$.

The two portfolios are equal if $\Delta S_0 u - f_u$
$= \Delta S_0 d - f_d$

$$\therefore \quad \Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

so that the portfolio is risk-free and hence must earn the risk free rate of interest.

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The cost of the portfolio is $\Delta S_0 - f$

Since the portfolio grows at a risk free rate, it follows that:

$$(\Delta S_0 u - f_u)\, e^{-rT} = \Delta S_0 - f$$

Substituting Δ from the earlier equation simplifies to:

$$f = e^{-rT}\left[p f_u + (1-p) f_d\right]$$

$$\text{where, } p = \frac{\left[e^{rT} - d\right]}{u-d}$$

(ii) The option pricing formula does not involve probabilities of stock going up or down although it is natural to assume that the probability of an upward movement in stock increases the value of call option and the value of put option decreases when the probability of stock price goes down.

This is because we are calculating the value of option not in absolute terms but in terms of the value of the underlying stock where the probabilities of future movements (up and down) in the stock already incorporates in the price of the stock. However, it is natural to interpret p as the probability of an up movement in the stock price.

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(iii) The expected stock price $E(S_T)$ at time $T = p S_0 u + (1-p) S_0 d$ 0.5

Substituting p from above equation (i) i.e.

$$p = \frac{(e^{rT} - d)}{(u-d)} \quad \text{—— (1)}$$

We get,

$$E(S_T) = e^{rT} S_0$$

i.e Stock price grows at a risk free rate or return on a stock is risk free rate.

(iv) In a risk neutral word individuals do not require compensation for risk or they are indifferent to risk. Hence expected return on all securities and options is the risk neutral discounted at risk free rate.

Q9.

(i) The forward price is given by
$f = S \cdot \exp(rt)$ where S is the stock price, t is the delivery time and r is the continuously compounded risk-free rate of interest applicable up to time t.

Put-call parity states that: $c + k \cdot \exp(-rt) = p + S$
where c and p are the prices of a European call and put option respectively with strike k and

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time to expiry t and S is the current stock price.

To compute F, we need to find S and r, t is given to be 0.25 years.

Sub. the values from the first two rows of the table in the put-call parity, we get two equations in two unknown (S and r:)

$$13.334 + 70 \cdot e^{-0.25r} = 0.120 + S$$
$$8.869 + 75\, e^{-0.25r} = 0.568 + S$$

Solving :- for S and r
$S = 82$ and $r = 7\%$

$\therefore$ We get the forward price $F = 82\, e^{0.07 \times 0.25}$
$= 83.45$.

ii) Let the continuously compounded, annualized) rate of interest over the next k months be r_k Then the required forward rate r_F can be found from:-

$$e^{r_6 \times 0.5} = e^{r_3 \times 0.25}\, e^{r_F \times 0.25}$$

$r_3 = 7\%$

Substitute values from the last row $S = 82$;

$$2.569 + 90\, e^{-0.5 \times r_6} = 7.909 + 82$$
$\therefore$ $r_6 = 6\%$ and $r_F = 5\%$

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(iii) Using the put-call parity for each row in the given table, we get:-

$$6.899 + a\exp(-0.07\times0.25) = 1.055 + 82$$
$$b + 80e^{-0.07\times0.25} = 1.789 + 82$$
$$2.594 + 85e^{-0.07\times0.25} = c + 82$$

$$a = 77.5 \quad , b = 5.177 \quad , c = 4.119$$

Q10.

(i) Since interest rates are assumed zero, the risk-neutral up-step probability is given as:-

$$q = \frac{1-d}{u-d}$$

where u and d are the sizes of up-step and down-step respectively

For a recombining tree, $d = \frac{1}{u}$

Subst. $d = \frac{1}{u}$

$$q = \frac{1-1/u}{u-1/u}$$
$$= \frac{1}{(u+1)}$$

For no-arbitrage to hold, we must have $u > 1 > d$

Then, $u > 1 \Rightarrow u+1 > 2$

$$q = \frac{1}{(u+1)} < 1/2$$

Hence proved

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(ii) As derived for part (a), $q = \frac{1}{u+1}$

Subs. t $q = \frac{1}{3}$ and solving for u we get; $u = 2$

For the calibration of a recombing binomial tree, we know that $u = \exp(\sigma . \sqrt{\Delta t})$, we can solve using $u = 2$ and $\Delta t = \frac{1}{12}$, we can solve for

$\sigma = 240.11\%$

Q12 Given

$z(t)$ is standard Brownian

(a) $d u(t) = 2 d z(t) - 0$
$= 0 \, dt + 2 d z(t)$

$\therefore$ The stochastic process $\{u(t)\}$ has zero drift.

(b) ~~d (v)~~ $d\, v(t) = d\, [z(t)]^2 - dt$; $2/2 [d z(t)]^2$
$d [z(t)^2] = 2 z(t) \, d z(t) + 2/2 [d z(t)]^2$
$= 2 z(t) \, d z(t) + dt$ by the multiplication rule.

(c) $d\, w(t) = d\, [t^2 z(t)] - 2t \, z(t)$
Because $d [t^2 z(t)] = t^2 \, d z(t) + 2t \, z(t) dt$,
$d\, w(t) = t^2 \, d z(t)$

$\therefore$ The process $w\{(t)\}$ has zero drift.

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Q13.

(1) Let S_t / S_0 follows lognormal distribution with parameters $(\mu - \frac{1}{2}\sigma^2)t$, and $\sigma^2 t$ such the expected return on a stock is μ and volatility is σ.

This means Expected value of stock price at the end of first time step $= S_0 e^{\mu \delta t}$. On the tree the expected price at that time

$$= q S_0 u + (1-q) S_0 d .$$

In order to match the expected return on the stock with the tree's parameters we have.

$$q S_0 u + (1-q) S_0 d = S_0 \mu^{u \delta t}$$

Volatility σ of a stock price is defined so that $\sigma\sqrt{\delta t}$ is the standard deviation of the return on the stock price in a short period of time δt.

The variance of stock price return is

$$q u^2 + (1-q) d^2 - [qu + (1-q)d]^2 = \sigma^2 \delta t$$

$$e^{\mu \delta t}(u+d) - ud - e^{2\mu \delta t} = \sigma^2 \delta t$$

when higher powers of δt other than δt are ignored. This implies $u = e^{\sigma\sqrt{\delta t}}$ and

$$d = \frac{1}{u} = e^{-\sigma\sqrt{\delta t}}$$