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PROB. & STATISTICS

ASSIGNMENT - 2

Q 1 ⇒

Let X be the amount of a home insurance claim.
Let Y be the amount of a car insurance claim.

$X \sim N(800, 100^2)$
$Y \sim N(1200, 300^2)$

Required probability:

$P((Y_1+Y_2+Y_3) > (X_1+X_2+X_3+X_4)+800)$
$= P((Y_1+X_2+Y_3) - (X_1+X_2+X_3+X_4) > 800)$

Distribution of $((Y_1+Y_2+X_3) - (X_1+X_2+X_3+Y_4))$
$(Y_1+Y_2+Y_3)-(X_1+X_2+X_3+X_4) \sim N((3 \times 1200)-(4 \times 800);$
$3 \times 300^2 + 4 \times 100^2)$

$(Y_1+Y_2+Y_3)-(X_1+X_2+X_3+X_4) \sim N(400, 310000)$

Thus,

$P((Y_1+Y_2+Y_3)-(X_1+X_2+X_3+X_4) > 800) = P\left(Z > \frac{800-400}{\sqrt{310000}}\right)$

$= P(Z > 0.71842)$
$= 1 - P(Z < 0.71842)$

$P(Z < 0.71) = 0.76115$
$P(Z < 0.72) = 0.76424$
0.01 causes change of $= 0.00309$

$$0.00842 = \frac{0.00309 \times 0.00842}{0.01}$$

$$P(Z < 0.71842) = 0.76375$$

$$P((Y_1+Y_2+Y_3) - (X_1+X_2+X_3+X_4) > 800) = 1 - P(Z < .71842)$$

$$= 1 - 0.76375$$

$$= 0.23625$$

Ques 2 ⇒

$$P(N=n) = {}^{n}C_n \; 0.9^{3} \; 0.1^{n}$$

$$p = 0.9$$

$$q = 0.1$$

$$k = 3$$

$$M_N(t) = \left(\frac{pe^t}{1-qe^t}\right)^k$$

$$= \left(\frac{0.9e^t}{1-0.1e^t}\right)^3$$

Let claim amt be $= X$

$X \sim$ Gamma $(6, 2)$

$$M_X(t) = \left(1 - \frac{t}{2}\right)^{-6}$$

(i) $M_Y(t) = M_N(\log M_X(t))$

$= M_N\left(\log (1-t/2)^{-6}\right)$

$= M_N\left(-6 \log (1-t/2)\right)$

$= \left(\dfrac{0.9 e^{\log (1-t/2)^{-6}}}{1-0.1 e^{\log (1-t/2)^{-6}}}\right)^3$

$M_Y(t) = \left(\dfrac{0.9 (1-t/2)^{-6}}{1-0.1(1-t/2)^{-6}}\right)^3$

(ii) $E(N) = \dfrac{k}{p} = \dfrac{3}{0.9} = 30/9 = 10/3$

$V(N) = \dfrac{kq}{p^2} = \dfrac{3(0.1)}{0.9 \times 0.9} = \dfrac{10}{27}$

$E[X] = \dfrac{\alpha}{\lambda} = \dfrac{6000}{2000} = 3$

$V[X] = \dfrac{\alpha}{\lambda^2} = \dfrac{3}{2000}$

$V(Y) = E(N)V(X) + V(N)(E(X))^2$

$= \dfrac{10}{3} \times \dfrac{3}{2000} + \dfrac{10}{27} \times 9$

$V(Y) = 3.338$

Let standard deviation be δ.

$$\delta(Y) = \sqrt{V(Y)}$$

$$\delta(Y) = 1.82711$$

Ques 3 (1)

$$M_X(t) = E(e^{tX})$$

$$= \int_{\alpha}^{\infty} e^{tx} \lambda e^{-\lambda(x-\alpha)} dx$$

$$= \lambda e^{\lambda\alpha} \int_{\alpha}^{\infty} e^{x(t-\lambda)}$$

$$= \frac{\lambda e^{\lambda\alpha}}{t-\lambda} \left[e^{-x(-t+\lambda)} \right]_{\alpha}^{\infty}$$

$$= \frac{\lambda e^{\alpha\lambda}}{-(t-\lambda)} \left[e^{-\infty} - e^{-\alpha(\lambda-t)} \right]$$

$$= \frac{\lambda e^{\alpha t}}{\lambda - t}$$

(iii) $m'_x(t) = \lambda \frac{d}{dt} \frac{e^{t\alpha}}{\lambda - t}$

$$= \lambda \left\{ \frac{\alpha e^{t\alpha}(\lambda - t) + e^{t\alpha}}{(\lambda - t)^2} \right\}$$

$m'_x(0) = E(X)$

$$E[X] = \lambda \left\{ \frac{\alpha e^{0}(\lambda) + e^{0}}{\lambda^2} \right\}$$

$$= \frac{\lambda\{\lambda\alpha + 1)}{\lambda}$$

$$= \frac{\lambda\alpha + 1}{\lambda}$$

$$E[X] = \alpha + \frac{1}{\lambda}$$

$$m''_x(t) = \lambda \frac{d}{dt} \left\{ \frac{\alpha e^{t\alpha}(\lambda - t) + e^{t\alpha}}{(\lambda - t)^2} \right\}$$

$$= \lambda \left\{ \frac{(\alpha^2 e^{t\alpha}(\lambda - t) - \alpha e^{t\alpha} + \alpha e^{t\alpha})(\lambda - t)^2 + 2(\lambda - t)(\alpha e^{t\alpha}(\lambda - t) + e^{t\alpha})}{(\lambda - t)} \right.$$

$m''_x(0) = E[X^2]$

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$$E(X^2) = \frac{1}{\lambda^3}\{\alpha^2\lambda^3 + 2\lambda^2\alpha + 2\lambda\}$$

$$E(X^2) = \alpha^2 + \frac{2\alpha}{\lambda} + \frac{2}{\lambda^2}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$= \alpha^2 + \frac{2\alpha}{\lambda} + \frac{2}{\lambda^2} - \alpha^2 - \frac{1}{\lambda^2} - \frac{2\alpha}{\lambda}$$

$$V(X) = \frac{1}{\lambda^2}$$

Ques 4 =>

$$G_V(t) = p^k(1-qt)^{-k}$$

$$G_V(t) = E(t^V)$$

$$= t^0 P(V=0) + t^1 P(V=1) + t^2 P(V=2) + \cdots$$

$$G'_V(0) = P(V=1)$$

$$G''_V(0) = P(V=2)$$

$$G'''_V(0) = P(V=3)$$

$$G''''_V(0) = P(V=4)$$

$$G'_V(t) = p^k(1-qt)^{-k-1} \times (-k) \times (-q)$$

$$= kq\,p^k(1-qt)^{-k-1}$$

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$$\psi''V(t) = kqp^k\,(-(k+1))\,(1-qt)^{-k-2}(-q)$$
$$= p^k q^2 k(k+1)(1-qt)^{-(k+2)}$$

$$\psi'''V(t) = p^k q^2 k(k+1)(-(k+2))(1-qt)^{-(k+3)}(-q)$$
$$= p^k q^3 k(k+1)(k+2)(1-qt)^{-(k+3)}$$

$\vdots$

$$\psi''''V(t) = p^k q^4 k(k+1)(k+2)(k+3)$$
$$P(V=4) = p^k q^4 k(k+1)(k+2)(k+3)$$

Question 5 (i)

$$f(x,y) = ax(x+y)$$

$$\int_x \int_y a(x)(x+y)\,dy\,dx = 1$$

$$a\int_{x=0}^{5}\int_{y=10}^{20}(x^2+xy)\,dy\,dx = 1$$

$$a\int_{x=0}^{5}\left[x^2y+\frac{xy^2}{2}\right]_{10}^{20}dx = 1$$

$$a\int_{0}^{5}(10x^2+150x)\,dx = 1$$

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$$a\,[2291.666667] = 1$$

$$a = 0.000436364$$

$$E[Y/X] = \int_y y\,\frac{f(x,y)}{f_X(x)}\,dy$$

$$= \int_{10}^{20} \frac{y \times a(x+y)}{10 \times a(x+15)}\,dy$$

$$= \frac{1}{10(x+15)}\left[\frac{xy^2}{2} + \frac{y^3}{3}\right]_{10}^{20}$$

$$= \frac{1}{10(x+15)}\left[200x + \frac{8000}{3} - 50x - \frac{1000}{3}\right]$$

$$= \frac{1}{10(x+15)}\left[150x + \frac{7000}{3}\right]$$

$$= \frac{1}{10(x+15)}\left[\frac{450x + 7000}{3}\right]$$

$$E[Y/X] = \frac{450x + 7000}{30(x+15)}$$

$$E[Y/X] = \frac{45x + 700}{3x + 45}$$

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$\rightarrow \quad P(Y > \frac{1}{3}X + 10)$

$$f(y) = \int_X f(x,y)\,dx$$

$$= a\int_0^5 (x^2 + xy)\,dx$$

$$= a\left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_0^5$$

$$= a\left[\frac{125}{3} + \frac{25y}{2}\right]$$

$$= \frac{a}{6}[75y + 250]$$

when $x = 0$, $y = 10$

when $x = 5$, $y = 35/3$

$$P(Y > \tfrac{1}{3}(X+10)) = \int_{10}^{35/3} f(y)\,dy$$

$$= \frac{a}{6}\int_{10}^{35/3} (75y + 250)\,dy$$

$$= \frac{a}{6}\left[\frac{30625 + 17500 - 22500 - 15000}{6}\right]$$

$$= \frac{a}{36} \times 10625$$

$$P\left(Y > \tfrac{1}{3}X + 10\right) = 0.128788$$

Question 6.

Case I

$X \sim Poi(\lambda)$

$Y \sim Poi(\mu)$

$X - Y \sim ??$

Let $Z = X - Y$

$$M_Z(t) = E(e^{tZ})$$
$$= E(e^{tX - tY})$$
$$= E(e^{tX} e^{-tY})$$
$$= E(e^{tX}) \cdot E(e^{-tY})$$
$$= M_X(t) \cdot M_Y(-t)$$
$$= e^{\lambda(e^t - 1)} e^{\mu(e^{-t} - 1)}$$
$$= e^{\lambda(e^t - 1) + \mu(e^{-t} - 1)}$$

Thus,

$X - Y$ does not follow $Poi(\lambda - \mu)$

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Case II:

$X \sim Poi(\lambda)$

$Y \sim Poi(\mu)$

$X+Y \sim ??$

let $Z = X+Y$

$$M_Z(t) = E(e^{tZ})$$
$$= E(e^{t(X+Y)})$$
$$= E(e^{tX} e^{tY})$$
$$= E(e^{tX}) E(e^{tY})$$
$$= M_X(t) M_Y(t)$$
$$= e^{\lambda(e^t-1)} e^{\mu(e^t-1)}$$
$$= e^{\mu+\lambda(e^t-1)}$$

Thus,

$X+Y \sim Poi(\lambda+\mu)$

Question 7 (i) $Var(X/Y=2)$

$$V(X/Y=2) = E\left[X^2/Y=2\right] - E^2\left[X/Y=2\right]$$

$$E[X/Y=2] = \Sigma x\, P(X=x/Y=2)$$

$$= \Sigma x \frac{P(X=x, Y=2)}{P(Y=2)}$$

$$P(Y=2) = \sum_x P(X=x, y=2)$$
$$= 0 + 0.3 + 0.1 + 0.2$$

$$P(Y=2) = 0.6$$

$$E[X/y=2) = 1 \times \frac{0}{0.6} + 2 \times \frac{0.3}{0.6} + 3 \times \frac{0.1}{0.6} + 4 \times \frac{0.2}{0.6}$$

$$E[X/Y=2] = 17/6$$

$$E[X^2/Y=2] = \sum x^2 P(X=x/Y=2)$$

$$= \sum x^2 \frac{P(X=x, y=2)}{P(Y=2)}.$$

$$= \frac{12}{6} + 9/6 + \frac{32}{6} = \frac{53}{6}$$

$$\text{var}(X/Y=2) = \frac{53}{6} - \left(17/6\right)^2$$

$$= 53/6 - \frac{289}{36}$$

$$\text{Var}(X/Y=2) = \frac{29}{36} = 0.80556$$

(II) $E(U/V=v) = \int_u u\, f(U/V)\, du$

$$f(v) = \int_0^1 \frac{48}{67}(20v - u^2)\, du$$

$$= \frac{48}{67}\left[u^2 v - \frac{1}{3} v^3\right]_0^1$$

$$= \frac{16(3v-1)}{67}$$

$$f(u/v) = \frac{f(u,v)}{f(v)}$$

$$= \frac{48}{67}(20v - v^2) \times \frac{67}{16(3v-1)}$$

$$= \frac{2uv - u^2}{v - 1/3}$$

$$E(U/V=v) = \int_0^1 \frac{2u^2 v - u^3}{v - 1/3}\, du$$

$$= \frac{3}{3v-1}\left[\frac{2}{3}u^3 v - \frac{u^4}{4}\right]_0^1$$

$$= \frac{3}{3v-1}\left(\frac{8v-3}{4}\right)$$

$$E(U/V=v) = \frac{8v-3}{4(3v-1)}$$

Ques 8

$$X \sim \text{Gamma}(3,2)$$

$$E(X) = \frac{\alpha}{\lambda} = 3/2 = 1.5$$

$$V(X) = \frac{\alpha}{\lambda^2} = \frac{3}{4} = 0.75$$

$$E(Y) = E\left[E[Y/X]\right]$$

$$= E(3X+1)$$

$$= 3E[X]+1$$

$$= 3(1.5)+1$$

$$E[Y] = 5.5$$

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$$V(Y) = V(E(Y/X)) + E(V(Y/X))$$
$$= E(2x^2+5) + V(3x+1)$$
$$= 2E(x^2) + 5 + 9V(x)$$

$$V(x) = E[X^2] - [E(X)]^2$$
$$E[X^2] = 0.75 + (1.5)^2$$
$$= 3$$

$$V(Y) = 2(3) + 5 + 9(0.75)$$
$$= 11 + 6.75$$
$$V(Y) = 17.75$$

$$S.D. \text{ of } Y = \sqrt{\frac{V(Y)}{17.75}} = 4.21307$$

Ques 9

(i) $$COV(X,Z) = COV(X, X+Y)$$
$$= COV(X,X) + COV(X,Y)$$
$$= V(X) + COV(X,Y)$$

$$corr(X,Y) = -0.3$$

$$COV(X,Y) = -0.6$$

$$COV(X,Z) = V(X) + COV(X,Y)$$
$$= 4 - 0.6$$

 $$COV(X,Z) = 3.4$$

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ii) $V(Z) = cov(Z, Z)$

$= cov(X+Y, X+Y)$

$= cov(X, X) + cov(X, Y) + cov(X, Y) + cov(Y, Y)$

$= V(X) + V(Y) + 2cov(X, Y)$

$= 4 + 1 - 2(0.6)$

$= 5 - 1.2$

$V(Z) = 3.8$

Question 10

$f(x, y) = k e^{-\alpha} e^{-y/\beta}$

$\int_X \int_Y f(x, y)\, dx\, dy = 1$

$\int_{x=1}^{\infty} \int_{y=1}^{\infty} k x^{-\alpha} e^{-y/\beta}\, dy\, dx = 1$

$k \int_{1}^{\infty} x^{-\alpha} \left[\beta e^{-y/\beta}\right] dx = 1$

$k \int_{1}^{\infty} x^{-\alpha} (\beta e^{-1/\beta})\, dx = 1$

$k\beta e^{-1/\beta} \int_{1}^{\infty} x^{-\alpha}\, dx = 1$

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$$\beta k e^{-1/\beta} \left[\frac{x^{-\alpha+1}}{-\alpha+1} \right]_1^{\infty} = 1$$

$$\beta k e^{-1/\beta} \left[\frac{1}{\alpha-1} - 0 \right] = 1$$

$$k = \frac{\alpha-1}{\beta e^{-1/\beta}} = k = \frac{(\alpha-1)}{\beta} e^{1/\beta}$$