

Financial Engineering Assignment 1

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1)

i)

a. Arbitrage opportunity is a situation where we can make a certain profit with no risk. This is sometimes described as a free lunch.

b. Arbitrage exists to exploit the mispricing/inefficiency in the markets.

c. It is exploitable and fleeting in nature.

d. There are two types of arbitrages:

Type I arbitrage – Immediate profit with zero probability of future loss.

Type II arbitrage – Zero cost with zero probability of future loss and some non-zero probability of future profit.

ii) The Law of one price states that any two portfolios that behave in exactly the same way must have the same price. If this were not true, we could buy the 'cheap' one and sell the 'expensive' one to make an arbitrage (risk- free) profit.

(iii)

Given: 3 month European put option

$S_0 = \text{Rs } 125$ and $K = \text{Rs } 120$

$c_t = \text{Rs } 30$ and $r = 5\% \text{ p.a.}$

(a) $q = 15\% \text{ p.a.}$

Using put call parity,

$$p_t + S_0 e^{-qt} = c_t + K e^{-rt}$$

$$p_t = 30 + 120 e^{-(5\% \cdot 3/12)} - 125 e^{-(15\% \cdot 3/12)} = \text{Rs } 28.11$$

(b) Arbitrage opportunity if put option price is Rs 23

The put option is cheap as it is selling at Rs 23 whereas as per the put-call parity, it should be Rs 28.11. Thus, we buy the cheap side and sell the other side i.e., buy 1 put option and share and sell 1 call option and borrow (sell) cash.

2) Let $dX_t = A_t dt + B_t dZ_t$,

Where, $A_t = \alpha\mu(T-t)$, $B_t = \sigma\sqrt{(T-t)}$

$$dF = \frac{\partial f}{\partial x} B_t dZ_t + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} A_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} B_t^2 \right) dt \quad (\text{Ito's Lemma})$$

$$dF = -f B_t dZ_t + f \frac{\partial m}{\partial t} (T-t) dt - f A_t dt + \frac{1}{2} f B_t^2 dt$$

$$\left[\text{Since } \frac{\partial f}{\partial x} = -e^{m(T-t)-x} \text{ and } \frac{\partial f}{\partial x} = \frac{\partial m}{\partial t} (T-t) * e^{m(T-t)-x} \text{ (using chain rule) and } \frac{\partial^2 f}{\partial x^2} = e^{m(T-t)-x} \right]$$

$$dF = f \left(\frac{\partial m}{\partial t} (T-t) - A_t + \frac{1}{2} B_t^2 \right) dt - f B_t dZ_t$$

For f to be a martingale, $\frac{\partial m}{\partial t} (T-t) - A_t + \frac{1}{2} B_t^2 = 0$

Thus, $\frac{\partial m}{\partial t} (T-t) = A_t - \frac{1}{2} B_t^2$

Substituting Eq 1 above gives $\frac{\partial m}{\partial t} = \alpha\mu - \frac{1}{2} \sigma^2$

3)

(i) Given: $f(t, x) = e^{-t} * x^2$

$$\frac{df}{dt} = -e^{-t} * x^2 = -f$$

Now, $\frac{df}{dx} = 2e^{-t} * x$

$$\frac{df}{dx^2} = 2e^{-t}$$

Using Ito's lemma,

$$dY_t = \frac{df}{dt} * dt + \frac{df}{dx} * dX_t + \frac{1}{2} * \frac{d^2 f}{dx^2} * \sigma^2 X_t^2 dt$$

$$dY_t = -f dt + 2 * e^{-t} X_t dX_t + e^{-t} * \sigma^2 X_t^2 dt$$

$$dY_t = -Y_t dt + 2 * e^{-t} X_t^2 * \frac{dX_t}{X_t} + e^{-t} \sigma^2 X_t^2 dt$$

$$dY_t = -Y_t dt + 2 * Y_t(0.25dt + \sigma * dW_t) + \sigma^2 Y_t dt$$

Taking Y_t out common,

$$\frac{dY_t}{Y_t} = (2 * 0.25 - 1 + \sigma^2)dt + 2\sigma dW_t$$

$$\frac{dY_t}{Y_t} = (\sigma^2 - 0.5)dt + 2\sigma dW_t$$

Therefore,

$$dY_t = (\sigma^2 - 0.5)Y_t dt + 2\sigma Y_t dW_t$$

(ii) The process is a martingale only if the drift is 0. Thus,

$$\sigma^2 - 0.5 = 0$$

$$\sigma^2 = 0.5$$

- 4) Let n ex-dividend dates are anticipated for a stock and $t_1 < t_2 < \dots < t_n$ are the times before which the stock goes ex-dividend. Dividends are denoted by $d_1, \dots, d_n$.

If the option is exercised prior to the ex-dividend date then the investor receives $S(t_n) - K$. If the option is not exercised, the price drops to $S(t_n) - d_n$.

The value of the American option is greater than $S(t_n) - d_n - K \exp(-r(T-t_n))$.

It is never optimal to exercise the option if $S(t_n) - d_n - K \exp(-r(T-t_n)) \geq S(t_n) - K$ i.e., $d_n \leq K(1 - \exp(-r(T-t_n)))$.

Using this equation: we have $K(1 - \exp(-r(T-t_n))) = 350(1 - \exp(-0.95(0.8333 - 0.25))) = 18.87$ and $65(1 - \exp(-0.95(0.8333 - 0.25))) = 10.91$. Hence it is never optimal to exercise the American option on the two ex-dividend rates.

- 5)

$$\frac{\ln S_t - \mu(S_t)}{\frac{1}{\sigma t^2}} \text{ follows standard normal distribution}$$

The probability that stock price will be higher than the strike price of Rs 258 in 6 months' time = $1 - N(5.55, 5.59)/0.247 = 0.5542 \dots$ (from Tables)

The put option is exercised if the stock price is less than Rs 258 in 6 months' time. The probability of this event = $1 - 0.5542 = 0.4457$

6)

(i) The given relationship can be written as:

$$S_t = S_0 e^{\mu t + \sigma B_t}$$

Since S_t is a function of standard Brownian motion, B_t , applying Ito's Lemma, the SDE for the underlying stochastic process becomes:

$$dB_t = 0 \times dt + 1 \times dB_t$$

Let $G(t, B_t) = S_t = S_0 e^{\mu t + \sigma B_t}$, then

$$\frac{dG}{dt} = \mu S_0 e^{\mu t + \sigma B_t} = \mu S_t$$

$$\frac{dG}{dB_t} = \sigma S_0 e^{\mu t + \sigma B_t} = \sigma S_t$$

$$\frac{d^2G}{dB_t^2} = \sigma^2 S_0 e^{\mu t + \sigma B_t} = \sigma^2 S_t$$

Hence, using Ito's Lemma from Page 46 in the Tables we have:

$$dG = [0 \times \sigma S_t + \frac{1}{2} \times 1^2 \times \sigma^2 S_t + \mu S_t]dt + 1 \times \sigma S_t dB_t$$

$$\text{i.e. } dS_t = (\mu + \frac{1}{2} \sigma^2) S_t dt + \sigma S_t dB_t$$

Thus,

$$\frac{dS_t}{S_t} = \sigma dB_t + (\mu + \frac{1}{2} \sigma^2) dt$$

$$\text{So, } c_1 = \sigma \text{ and } c_2 = \mu + \frac{1}{2} \sigma^2$$

(ii) The expected value of S_t is:

$$E[S_t] = E[S_0 e^{\mu t + \sigma B_t}] = S_0 e^{\mu t} E[e^{\sigma B_t}]$$

$$\text{Since } B_t \sim N(0,1), \text{ its MGF is } E[e^{\theta B_t}] = e^{\frac{1}{2}\theta^2 t}$$

$$\text{So, } E[S_t] = S_0 e^{\mu t} \times e^{\frac{1}{2}\sigma^2 t} = S_0 e^{\mu t + \frac{1}{2}\sigma^2 t}$$

The variance of S_t is:

$$\begin{aligned}\text{Var}[S_t] &= E[S_t^2] - (E[S_t])^2 = E[S_0^2 e^{2\mu t + 2\sigma B_t}] - (S_0 e^{\mu t + \frac{1}{2}\sigma^2 t})^2 \\ &= S_0^2 e^{2\mu t} E[e^{2\sigma B_t}] - S_0^2 e^{2\mu t} + \sigma^2 t = S_0^2 e^{2\mu t + 2\sigma^2 t} - S_0^2 e^{2\mu t + \sigma^2 t} \\ &= S_0^2 e^{2\mu t} (e^{\sigma^2 t} - e^{\frac{1}{2}\sigma^2 t})\end{aligned}$$

(iii)

The covariance –

$$\text{Cov}[S_{t_1}, S_{t_2}] = E[S_{t_1} S_{t_2}] - E[S_{t_1}] E[S_{t_2}]$$

Solving each term,

$$E[S_{(t_1)}] = S_0 * e^{(\mu * t_1 + 1/2 \sigma^2 t_1)}$$

$$E[S_{(t_2)}] = S_0 * e^{(\mu * t_2 + 1/2 \sigma^2 t_2)}$$

$$\begin{aligned}E[S_{(t_1)} S_{(t_2)}] &= E[S_0 * \exp(\mu t_1 + \sigma B_{(t_1)}), S_0 \exp(\mu t_2 + \sigma B_{(t_2)})] \\ &= S_0^2 * e^{(\mu(t_1 + t_2))} E[\exp(\sigma B_{(t_1)} + \sigma B_{(t_2)})]\end{aligned}$$

Splitting $B_{(t_2)}$ into two independent components,

$$B_{(t_2)} = B_{(t_1)} + (B_{(t_2)} - B_{(t_1)}) \text{ where } B_{(t_2)} - B_{(t_1)} \sim N(0, t_2 - t_1)$$

Hence,

$$E[S_{(t_1)} S_{(t_2)}] = S_0^2 * e^{\mu(t_1 + t_2)} * e^{(3/2 \sigma^2 t_1 + 1/2 \sigma^2 t_2)}$$

Substituting in the formula of covariance,

$$\begin{aligned}\text{Cov}[S_{(t_1)}, S_{(t_2)}] &= S_0^2 * e^{\mu(t_1 + t_2)} * [e^{(3/2 \sigma^2 t_1)} - e^{(1/2 \sigma^2 t_1)}] \\ &\quad * e^{(1/2 \sigma^2 t_2)}\end{aligned}$$

7)

(i)

Constructing the binomial tree with 'u' as up move and 'd' as down move. Considering 'p' as the probability of up move.

Thus, the probability of down move is $(1 - d)$.

Then $E(C_t) = S_0[pu + (1-p)d]$, and

$$\text{Var}(C_t) = E(C_t^2) - E(C_t)^2$$

$$= S_0^2 [pu^2 + (1-p)d^2] - S_0^2 [pu + (1-p)d]^2$$

$$= S_0^2 [pu^2 + (1-p)d^2 - (pu + (1-p)d)^2]$$

$$= S_0^2 [p(1-p)u^2 + p(1-p)d^2 - 2p(1-p)] \dots (\because d = 1/u)$$

$$= S_0^2 p(1-p)(u-d)^2$$

Equating moments:

$$S_0 e^{rt} = S_0 [pu + (1-p)d] \quad (A)$$

And

$$\sigma^2 S_0^2 t = S_0^2 p(1-p)(u-d)^2 \quad (B)$$

$$\text{From (A) we get } p = (e^{rt} - d) / (u - d) \quad (C)$$

Substituting p into equation (B), we get

$$\sigma^2 t = (e^{rt} - d) / (u - d) (1 - e^{rt} - d) / (u - d) (u - d)^2$$

$$= - (e^{rt} - d) (e^{rt} - u)$$

$$= (u + d) e^{rt} - (1 + e^{2rt})$$

Putting $d = 1/u$, and multiplying through by u we get

$$u^2 e^{rt} - u (1 + e^{2rt} + \sigma^2 t) + e^{rt} = 0$$

This is a quadratic in u which can be solved in the usual way.

(ii)

(a)

The given and calculated data is $\sigma = 0.15$, $t = 0.25$ and $u = 1.077884$, $d = 0.92774$

$t=0$	$t=.25$	$t=.5$	$t=.75$	
			100.186	Node A
		92.947		
	86.231		86.232	Node B
80		80.001		
	74.22		74.22	Node C
		68.857		
			63.882	Node D

(b)

$r = 0$ and p using the formula is 0.48126

t=0	t=.25	t=.5	t=.75	
			20.186	Node A
		12.948		
	7.787		6.232	Node B
4.496		2.999		
	1.443		0	Node C
		0		
			0	Node D

Hence, the value of the call option is 4.496

(c)

The lookback call pays the difference between the minimum value and the final value. Notate paths by U for up and D for down, in order

We get the payoffs

UUU (100.186 - 80) = 20.186 Node A

UDU (86.232-80)= 6.232 Node B

UUD (86.232- 80) =6.232 Node B

UDD (74.22-74.22) = 0 Node C

DUU (86.232- 74.22) = 12.012 Node B

DUD (74.22-74.22) = 0 Node C

DDU (74.22-68.857) = 5.363 Node C

DDD (63.882-63.882)= 0 Node D

The lookback payoffs are, for each successful path (i.e., with a non- zero result) Probabilities of arriving at each node are:

Node A= $p^3 = 0.11147$

Node B= $p^2(1-p) = 0.12015$

Node C= $p(1-p)^2 = 0.12950$

Node D= $p(1-p)^3 = 0.13959$

Hence the tree value of lookback option is:

$$(0.11147 \times 20.186) + (0.12015 \times [6.232 + 6.232 + 12.012]) + (0.12950 \times 5.363) = 5.8854$$

8)

(i) Consider a stock whose current price is S_0 and an option whose current price is f . We suppose that the option lasts for time T and that during the life of the option the stock price can either move up from S_0 to a new level S_{0u} or move down to S_{0d} where $u > 1$ and $d < 1$.

Let the payoff be f_u if the stock price becomes S_{0u} and f_d if stock price becomes S_{0d} .

Let us construct a portfolio which consists of a short position in the option and a long position in Δ shares. We calculate the value of Δ that makes the portfolio risk-free.

Now if there is an upward movement in the stock, the value of the portfolio becomes $\Delta S_{0u} - f_u$ and if there is a downward movement of stock, the value of the portfolio becomes $\Delta S_{0d} - f_d$

The two portfolios are equal if $\Delta S_{0u} - f_u = \Delta S_{0d} - f_d$

Or $\Delta = (f_u - f_d) / (S_{0u} - S_{0d})$ so that the portfolio is risk-free and hence must earn the risk-free rate of interest.

This means the present value of such a portfolio is $(\Delta S_{0u} - f_u) \exp(-rT)$

Where r is the risk-free rate of interest.

The cost of the portfolio is $\Delta S_0 - f$

Since the portfolio grows at a risk-free rate, it follows that

$$(\Delta S_{0u} - f_u) \exp(-rT) = \Delta S_0 - f$$

$$\text{or } f = \Delta S_0 - (\Delta S_{0u} - f_u) \exp(-rT)$$

Substituting Δ from the earlier equation simplifies to:

$$f = e^{-rT} [p f_u + (1-p) f_d] \text{ where } p = [e^{rT} - d] / (u - d)$$

(ii) The option pricing formula does not involve probabilities of stock going up or down although it is natural to assume that the probability of an upward movement in stock increases the value of call option and the value of put option decreases when the probability of stock price goes down

This is because we are calculating the value of option not in absolute terms but in terms of the value of the underlying stock where the probabilities of future movements (up and down) in the stock already incorporates in the price of the stock. However, it is natural to interpret p as the probability of an up movement in the stock price. The variable $1-p$ is then the probability of a down movement such that the above equation can be interpreted as that the value of option today is the expected future value discounted at the risk-free rate

(iii) The expected stock price $E(S_T)$ at time $T = pS_0u + (1-p)S_0d$ 0.5

or $E(S_T) = pS_0(u-d) + S_0d$ ---0.5

Substituting p from above equation in (i) i.e., $p = [e^{rT} - d]/[u - d]$ ---1

We get $E(S_T) = e^{rT}S_0$ ---0.5 -----1

i.e., stock price grows at a risk-free rate or return on a stock is risk free rate

iv) In a risk neutral world individuals do not require compensation for risk or they are indifferent to risk. Hence expected return on all securities and options is the risk-free interest rate. Hence value of an option is its expected payoff in a risk neutral discounted at risk free rate.

9)

(i) The forward price is given by $F = S * e^{rt}$ where S is the stock price, t is the delivery time and r is the continuously compounded risk-free rate of interest applicable up to time t .

Put-call parity states that:

$c + K * e^{-rt} = p + S$ where c and p are the prices of a European call and put option respectively with strike K and time to expiry t and S is the current stock price.

To compute F , we need to find S and r . t is given to be 0.25 years.

Substituting the values from the first two rows of the table in the put-call parity, we get two equations in two unknowns (S and r):

$$13.334 + 70 * e^{(-0.25r)} = 0.120 + S$$

$$8.869 + 75 * e^{(-0.25r)} = 0.568 + S$$

Solving the simultaneous equations for S and r , we get:

$$S = 82 \text{ and } r = 7\%$$

Therefore, we get the forward price $F = 82 * e^{(0.07 * 0.25)} = 83.45$

(ii) Let the (continuously compounded, annualized) rate of interest over the next k months be r_k . Then the required forward rate r_F can be found from:

$$\exp(r_6 * 0.5) = \exp(r_3 * 0.25) * \exp(r_F * 0.25) \text{ or } 2 * r_6 = r_3 + r_F$$

We know that $r_3 = 7\%$.

To find r_6 , we substitute values from the last row in the put-call parity relationship and $S = 82$:

$$2.569 + 90 * e^{(-0.5 * r_6)} = 7.909 + 82$$

Therefore, $r_6 = 6\%$ and $r_F = 5\%$

(iii) Using the put-call parity for each row in the given table, we get:

$$6.899 + a \cdot \exp(-0.07 \cdot 0.25) = 1.055 + 82$$

$$b + 80 \cdot \exp(-0.07 \cdot 0.25) = 1.789 + 82$$

$$2.594 + 85 \cdot \exp(-0.07 \cdot 0.25) = c + 82$$

Solving individually, we get:

$$a = 77.5 \quad b = 5.177 \quad c = 4.119$$

10)

i) Since interest rates are assumed zero, the risk-neutral up-step probability is given as:

$$q = (1 - d) / (u - d)$$

where u and d are the sizes of up-step and down-step respectively

For a recombining tree, $d = 1/u$.

Substituting $d = 1/u$ in the expression for q and simplifying, we get:

$$q = (1 - 1/u) / (u - 1/u) = 1 / (u + 1)$$

For no-arbitrage to hold, we must have $u > 1 > d$.

Then, $u > 1 \Rightarrow u + 1 > 2 \Rightarrow q = 1 / (u + 1) < 1/2$. Hence proved.

ii) Since each step is one month and the expiry of the derivative is one year from now.

Therefore, a 12-step recombining binomial tree needs to be created, i.e. $n = 12$.

Further, at time $T = 12$ months, the stock price will be $S_0 u^k$

d^{n-k} with risk-neutral probability

$\binom{n}{k} q^k (1-q)^{n-k}$ where q , the up-step probability is $1/3$, u , the up-step size is 2 , and $d = 1/u = 1/2$.

We know that the derivative has a payoff $\sqrt{\frac{S_T}{S_0}}$ at time $T = 12$ months.

Thus, the current price of that derivative is: $P = \sum_{k=0}^n \sqrt{\frac{S_T}{S_0}} \cdot \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k}$

Therefore, $P = \sum_{k=0}^n \sqrt{\frac{S_0 u^k d^{n-k}}{S_0}} \cdot \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k} = \sum_{k=0}^n \sqrt{u^k d^{n-k}} \cdot \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k}$

$$P = \sum_{k=0}^n u^{\frac{k}{2}} d^{\frac{n-k}{2}} \cdot \frac{n!}{k!(n-k)!} q^k (1-q)^{n-k} = \sum_{k=0}^n 2^{\frac{k}{2}} \left(\frac{1}{2}\right)^{\frac{n-k}{2}} \cdot \frac{n!}{k!(n-k)!} \left(\frac{1}{3}\right)^k \left(\frac{2}{3}\right)^{n-k}$$

$$P = \sum_{k=0}^n 2^{k-\frac{n}{2}} \cdot \frac{n!}{k!(n-k)!} \frac{2^{n-k}}{3^n} = \sum_{k=0}^n 2^{\frac{n}{2}} \cdot \frac{n!}{k!(n-k)!} \frac{1}{3^n} = 2^{\frac{n}{2}} \frac{1}{3^n} \sum_{k=0}^n \frac{n!}{k!(n-k)!}$$

$$P = 2^{\frac{n}{2}} \frac{1}{3^n} 2^n = \left(\frac{2\sqrt{2}}{3}\right)^n = \left(\frac{2\sqrt{2}}{3}\right)^{12} = 0.49327$$

11)

(i) A recombining binominal tree or binominal lattice is one in which the sizes of the up-steps and down-steps are assumed to be the same under all states and across all time intervals. i.e., $u_t(j)=u$ and $d_t(j)=d$ for all times t and states j , with $d < \exp(r) < u$

1. It therefore follows that the risk neutral probability 'q' is also constant at all times and in all states e.g., $q_t(j)=q$

2. The main advantage of a 'n' period recombining binominal tree is that it has only [n+1] possible states of time as opposed to $2n$ possible states in a similar non-recombining binominal tree. This greatly reduces the amount of computation time required when using a binominal tree model.

3. The main dis-advantage is that the recombining binominal tree implicitly assumes that the volatility and drift parameters of the underlying asset price are constant over time, which assumption is contradicted by empirical evidence.

(ii)

(a) The risk-neutral probabilities at the first and second steps are as follows:

$$q_1 = (\exp(0.0175) - 0.95)/(1.10-0.95)$$

$$= (0.06765)/0.15$$

$$= 0.4510$$

$$q_2 = (\exp(0.025) - 0.90)/(1.20-0.90)$$

$$= 0.41772$$

Put payoffs at the expiration date at each of the four possible states of expiry are 0,0,0 and 95.

Working backwards, the value of the option $V_1(1)$ following an up step over the first 3 months is

$$V_1(1) \exp(0.025) = [0.41772 \cdot 0] + [0.58228 \cdot 0]$$

$$\text{i.e., } V_1(1) = 0$$

The value of the option $V_1(2)$ following a down step over the first 3 months is:

$$V_1(2) \exp(0.025) = [0.41772 \cdot 0] + [0.58228 \cdot 95]$$

$$\text{i.e., } V_1(2) = 53.9508$$

The current value of the put option is:

$$V_0 \exp(0.0175) = [0.4510 \cdot 0] + [0.5490 \cdot 53.9508]$$

$$\text{i.e., } V_0 = 29.105$$

(b) While the proposed modification would produce a more accurate valuation, there would be a lot more parameter values to specify. Appropriate values of u and d would be required for each branch of the tree and values of ' r ' for each month would be required.

The new tree would have $2^6 = 64$ nodes in the expiry column. This would render the calculations prohibitive to do normally, and would require more programming and calculation time on the computer.

An alternative model that might be more efficient numerically would be a 6-step recombining tree which would have only 7 nodes in the final column.

12)

Given $Z(t)$ is standard Brownian

$$\text{a) } dU(t) = 2dZ(t) \quad U(0) = 0 \Rightarrow dU(t) = 2dZ(t).$$

Thus, the stochastic process $\{U(t)\}$ has zero drift.

$$\text{b) } dV(t) = d[Z(t)]^2 \quad dt. \quad d[Z(t)]^2 = 2Z(t)dZ(t) + 2 \cdot \frac{1}{2} [dZ(t)]^2 = 2Z(t)dZ(t) + dt \text{ by the multiplication rule. Thus, } dV(t) = 2Z(t)dZ(t). \text{ The stochastic process } \{V(t)\} \text{ has zero drift.}$$

$$\text{c) } dW(t) = d[t^2 Z(t)] \quad dt. \quad \text{Because } d[t^2 Z(t)] = t^2 dZ(t) + 2tZ(t)dt, \text{ we have } dW(t) = t^2 dZ(t).$$

Thus, the process $\{W(t)\}$ has zero drift.

13)

Let S_t/S_0 follow lognormal distribution with parameters $(\mu - 1/2 \sigma^2)t$, and $\sigma^2 t$ such that the expected return on a stock is μ and volatility is σ .

This means expected value of stock price at the end of first-time step = $S_0 \cdot \exp(\mu \cdot \Delta t)$. The expected value after 1 time step is $qS_u + (1-q)S_d$

In order to match the expected return on the stock with the tree's parameters we have

$$q = (\exp(\mu \cdot \Delta t) - d) / (u - d)$$

Volatility σ of a stock price is defined so that $\sigma \cdot \sqrt{\Delta t}$ is the standard deviation of the return on the stock price in a short period of time Δt

Variance of stock price return is $\sigma^2 \cdot \Delta t$

Given data, $S = 200$, $r = 10\%$, $\sigma = 35\%$, $T = 2$ months, $t = 1$ month = $1/12 = 0.0833$

$u = 1.1063$, $d = 0.9039$, $q = 0.5161$, $(1 - q) = 0.4839$

$K = 200$

Using the binomial approach (Calculations in rough)

The value of put option = 6.05

14)

Given data: $S_0 = \text{Rs } 500$. Time step is of 3 months each. $u = 1.6$ and $d = 0.95$.

$r = 5\%$ p.a.

After constructing the binomial tree in rough and plotting the payoff wherein $S > K$,

The value of 6-month European call option = Rs 16.35

From the above constructed binomial tree, the payoff was changed and plotted as per put option where $K > S$.

The value of 6-month European put option = Rs 13.76

Using the put call parity

$$p + S_0 = c + K e^{-rt}$$

$$13.76 + 500 = 16.35 + 510 e^{(-5\% \cdot 0.5)}$$

The valuation of American option is somewhat similar to that of the European option except at the intermediary nodes we consider the expected and the exercise payoff and go ahead with whichever is the highest one.

After solving,

Value of American put option is Rs 16.46

Expected payoff in 3 months' time is calculated by using real rate of return of 9%. $p=0.6614$.
Hence expected payoff = $20 \times 0.6614 = 13.228$. Unfortunately, it is not easy to know the correct discount rate to apply to the expected payoff in real world to be able to compute the value of the option. Risk neutral valuation solves this problem as under risk neutral valuation all assets are expected to earn the risk-free rate.

15)

(i) Let $f = f(S_t, t) = S^k$

$$\frac{df}{ds} = kS^{k-1}, \frac{d^2f}{ds^2} = k(k-1) * S^{k-2}, \frac{df}{dt} = 0$$

$f = S^k$ follows Geometric brownian motion, with drift μ'

$$= k\mu + \frac{1}{2} * k * (k-1) * \sigma^2 \text{ and volatility } \sigma' = k * \sigma$$

$$\text{Hence, } f_t = f_0 * e^{(\mu * t - \frac{1}{2} \sigma^2 t) + \sigma W_t}$$

This means f_t/f_0 follows lognormal distribution.

(ii) Let $f = f(t, S_t) = e^{-rt} * S_t$

$$\frac{df}{dt} = -r * e^{-rt} * S_t, \frac{df}{ds} = e^{-rt}, \frac{d^2f}{ds^2} = 0$$

(iii)

Combining the results,

For given values of r , σ , and k , it can be solved for the value of μ for which discounted S^k will be a martingale.