1) Let X be claim size on home insurance
Y be claim size on car insurance.

$X \sim Normal(800, 100^2)$

$Y \sim Normal(1200, 300^2)$

$P(Y_1 + Y_2 + Y_3 > 800 + X_1 + X_2 + X_3 + X_4)$

$P((Y_1+Y_2+Y_3) - (X_1+X_2+X_3+X_4) > 800)$

$(Y_1+Y_2+Y_3) - (X_1+X_2+X_3+X_4) \sim N(3(1200) - 4(800), 3(300)^2 + 4(100^2))$

$\sim N(400, 310000)$

$P\left(Z > \frac{800-400}{\sqrt{310000}}\right) = P(Z > 0.71842121)$

$= 1 - P(Z < 0.7184212172)$

$= 1 - 0.76424$

$= 0.23576$

2) $N \sim$ Negative Bin (k, p)

$P(N=n) = {}^{n+2}C_{n-1} (0.9)^3 (0.1)^n$

$= {}^{(n+3)-1}C_{n-1} (0.9)^3 (0.1)^n$

$k = 3, \quad p = 0.9$

$M_Y t = M_N(\log M_X t)$

$X \sim Gamma(6, 2)$

Y = Total claim amount

i) MGF of Y :- $\quad M_Y(t) = E(E(e^{Yt}/N=n))$

$E(e^{Yt}/N=n) = E(e^{X_1t + X_2t + \cdots X_nt})$

$= E(e^{X_1t}) \cdot E(e^{X_2t}) \cdot E(e^{X_3t}) \cdots E(e^{X_nt})$

$= \left(1 - \frac{t}{2}\right)^{-6n}$

$E(E(e^{Yt}/N=n)) = E\left(1 - \frac{t}{2}\right)^{-6N} = E\left(\left(1 - \frac{t}{2}\right)^{-6N}\right)$

$= E(e^{N \log(1-\frac{t}{2})^{-6}})$

$= M_N\left(\log\left(1 - \frac{t}{2}\right)^{-6}\right)$

$$M_N(t) = \left(\frac{pe^t}{1-qe^t}\right)^k = \left(\frac{pe^{\log(1-\frac{t}{2})^{-6}}}{1-qe^{\log(1-\frac{t}{2})^{-6}}}\right)^k$$

$$= \frac{p(1-t/2)^{-6}}{1-q(1-t/2)^{-6}}$$

$$= \frac{0.9(1-t/2)^{-6}}{1-0.1(1-t/2)^{-6}}$$

$$V(Y) = E(N)V(X) + (E(X))^2 V(N)$$

$$= \left(\frac{3}{0.9} \times \frac{6}{4}\right) + \left(\frac{6}{2}\right)^2 \left(\frac{3\times 0.1}{(0.9)^2}\right)$$

$$= 8.333$$

$$\sigma = 2.8867$$

3) i) $f(x) = \lambda e^{-\lambda(x-\alpha)}$

$$MGF = E(e^{tX}) = \int_\alpha^\infty e^{tx} \lambda e^{-\lambda(x-\alpha)}\,dx$$

$$M_X(t) = \lambda \int_\alpha^\infty e^{tx} e^{-\lambda x} e^{\lambda\alpha}\,dx$$

$$= \lambda e^{\lambda\alpha} \int_\alpha^\infty e^{x(t-\lambda)}\,dx$$

$$= \lambda e^{\lambda\alpha} \left[\frac{e^{x(t-\lambda)}}{t-\lambda}\right]_\alpha^\infty$$

$$= \frac{\lambda e^{\lambda\alpha}}{t-\lambda}\left(-e^{-(\lambda-t)\alpha}\right)$$

$$= \frac{\lambda e^{t\alpha}}{\lambda - t}$$

ii) $M_X(t) = \dfrac{\lambda e^{t\alpha}}{\lambda - t} = \lambda e^{t\alpha}(\lambda-t)^{-1}$

$$M'_X(t) = \lambda\left[e^{t\alpha}(\lambda-t)^{-2} + (\lambda-t)^{-1}e^{t\alpha}(\alpha)\right]$$

$$= \lambda e^{t\alpha}(\lambda-t)^{-2}\left[\alpha(\lambda-t)+1\right]$$

$E(X)$ $t=0$, $M'_X(0) = \lambda^{-1}(\alpha\lambda+1)$

$$M''_X(t) = \lambda\left[\alpha e^{t\alpha}(\lambda-t)^{-2} + \alpha^2(\lambda-t)^{-1}e^{t\alpha} + e^{t\alpha}(2)(\lambda-t)^{-3} + \alpha(\lambda-t)^{-2}e^{t\alpha}\right]$$

$$M''_X(0) = \lambda\left[\alpha\lambda^{-2} + \alpha^2\lambda^{-1} + 2\lambda^{-3} + \alpha\lambda^{-2}\right]$$

$$E(X^2) = 2\alpha\lambda^{-1} + \alpha^2 + 2\lambda^{-2}$$

$$V(X) = E(X^2) - (E(X))^2$$
$$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \left(\alpha + \frac{1}{\lambda}\right)^2$$
$$= \frac{2\alpha}{\lambda} + \alpha^2 + \frac{2}{\lambda^2} - \alpha^2 - \frac{1}{\lambda^2} - \frac{2\alpha}{\lambda}$$
$$= \frac{1}{\lambda^2}$$

4) $G_V(t) = \left(\frac{p}{1-qt}\right)^k$

$= \Sigma t^v \times P(V=v)$

$P(V=4) = \frac{\Gamma k+4}{\Gamma 5 \; \Gamma k} p^k q^k$

$= \frac{\Gamma k+4}{\Gamma k \; 4!} p^k q^k$

$= \frac{(k+3)!}{(k-1)! \; 4!} p^k (1-p)^k$

5) $f(x,y) = ax(x+y) \qquad 0<x<5, \quad 10<y<20$

$P\left(Y > \frac{1}{3}x + 10\right) := a\int_{10}^{20}\int_0^5 (x^2+xy)\,dx\,dy = 1$

$a = ?$

$= \int_{10}^{20} \frac{x^3}{3} + \frac{x^2y}{2} \Big|_0^5 dy = \frac{1}{a}$

$= \int_{10}^{20} \frac{125}{3} + \frac{25y}{2}\,dy = \frac{1}{a}$

$= \frac{125y}{3} + \frac{25y^2}{4}\Big|_{10}^{20} = \frac{1}{a}$

$= \frac{1250}{3} + \frac{2500}{4} = \frac{1}{a}$

$a = 0.00096.$

6) $X \sim Poi(\lambda)$ $\quad Y \sim Pois(\mu)$

$M_X(t) = e^{\lambda(e^t-1)}$ $\quad M_Y(t) = e^{\mu(e^t-1)}$

$Z = X - Y$

$M_Z(t) = E(e^{tZ})$

$= E(e^{tX}) \cdot E(e^{-tY})$

$= \dfrac{e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}}$

$= e^{(\lambda-\mu)(e^t-1)}$

$\therefore \quad Z \sim Poi(\lambda-\mu)$

Similarly, $\quad Z = X+Y$

$E(e^{tZ}) = e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)}$

$= e^{(\lambda+\mu)(e^t-1)}$

$\therefore \quad Z \sim Poi(\lambda+r)$

10) $f(x,y) = k\,x^{-\alpha} e^{-y/B}$

$k \int_1^\infty \int_1^\infty x^{-\alpha} e^{-y/B} \, dx\,dy = 1$

$\dfrac{k \int e^{-y/B} \left[x^{-(\alpha-1)} \right]_1^\infty}{-\alpha+1} = 1$

$\therefore \dfrac{k}{1-\alpha} \int e^{-y/B} dy = 1$

$\therefore \dfrac{k}{(1-\alpha)} \dfrac{\left[e^{-y/B}\right]_1^\infty}{(-1/B)} = 1$

$\dfrac{-k e^{-1/B}(B)}{1-\alpha} = 1$

$k = \dfrac{\alpha-1}{B e^{-1/B}}$

9) $E(X) = 3$ $\quad V(X) = 4$

$E(Y) = 4$ $\quad V(Y) = 1$

Correlation coefficient $= -0.3\sqrt{4}$

$cov(X,Y) = -0.6$

$Z = X + Y$

i) $cov(X, X+Y) = cov(X,X) + cov(X,Y)$

$= V(X) + cov(X,Y)$

$= 4 + (-0.6) = 3.4$

ii) $var(Z) = cov(X+Y, X+Y)$

$= cov(X,X) + 2\,cov(X,Y) + cov(Y,Y)$

$= var(X) + V(Y) + 2\,cov(X,Y)$

$= 4 + 1 + 2(-0.6)$

$= 5 - 1.2 = 3.8$

8) $X \sim Gamma(3, 2)$

$E(X) = \frac{3}{2}$ $\quad V(X) = \frac{3}{4}$

$E(Y/X=x) = 3x+1$

$V(Y/X=x) = 2x^2+5$

$E(Y) = E(E(Y/X=x)) = E(3x+1)$

$= 3E(X) + 1$

$= \frac{9}{2} + 1 = \frac{11}{2}$

$V(Y) = E(V(\cancel{XY}\,Y/X)) + V(E(Y/X))$

$= E(2x^2+5) + V(3x+1)$

$= \cancel{2x}\ 2E(x^2) + 5 + 9V(X)$

$= 2\left(\frac{3}{4} + \frac{9}{4}\right) + 5 + 9\left(\frac{3}{4}\right)$

24 + 27 + 20 $\quad = \frac{71}{4}$