SRM Assignment 2

- Drashti Shah

Q1)

i. Writing the state space in order {Bid (B), Offer (O)} the generator matrix is:

$$\begin{array}{c} B \\ O \end{array} \begin{pmatrix} -\lambda & \lambda \\ \mu & -\mu \end{pmatrix}$$

ii. The holding times are exponentially distributed with parameter λ in state B, and μ in state O.

iii. $$\frac{\partial}{\partial t} \, {}_tP_s^{BB} = -\lambda \cdot {}_tP_s^{BB} + \mu \cdot {}_tP_s^{BO}$$

$$\frac{\partial}{\partial t} \, {}_tP_s^{BO} = \lambda \cdot {}_tP_s^{BB} - \mu \cdot {}_tP_s^{BO}$$

iv. We have two state model so:

$${}_tP_s^{BB} + {}_tP_s^{BO} = 1$$

substituting,

$$\frac{\partial}{\partial t}\left[\exp((\lambda+\mu)t) \cdot {}_tP_s^{BB}\right] = \mu \cdot \exp((\lambda+\mu)t);$$

and hence,

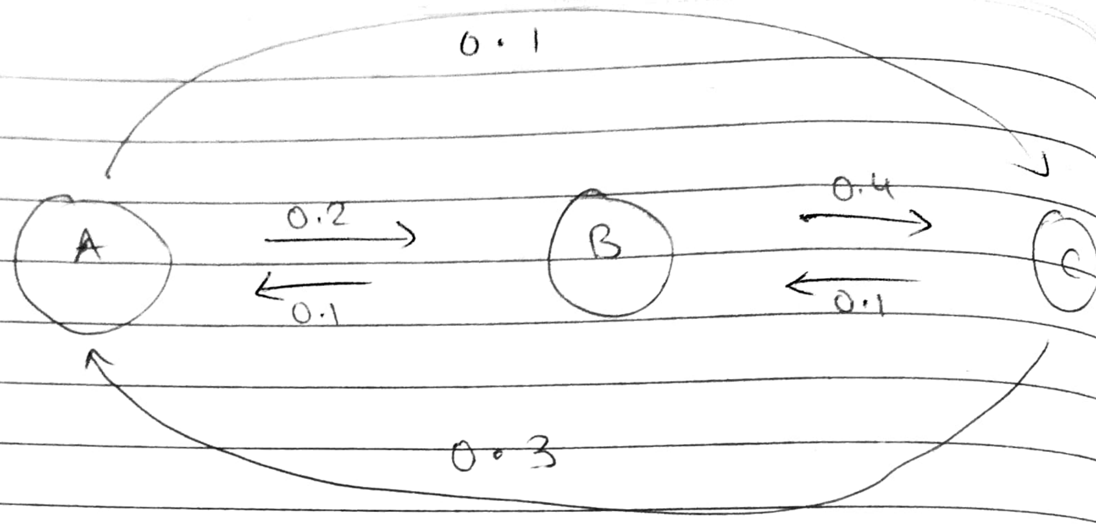
$$\exp((\lambda+\mu)t) \cdot {}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} \cdot \exp((\lambda+\mu)t) + \text{constant}.$$

Since the process is in state Bid at time s (ie. $t = 0$)

the constant is λ, & thus ${}_tP_s^{BB} = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \cdot \exp((\lambda + ...$

Q2)

i.



ii. $$\frac{d}{dt} P_{AA}(t) = -0.3 P_{AA}(t) + 0.1 P_{AB}(t) + 0.3 P_{AC}(t)$$

$$\frac{d}{dt} P_{AB}(t) = 0.2 P_{AA}(t) - 0.5_{AB}(t) + 0.1 P_{AC}(t)$$

$$\frac{d}{dt} P_{AC}(t) = 0.1 P_{AA}(t) + 0.4 P_{AB}(t) - 0.4 P_{AC}(t)$$

iii. we can model this an Poisson with parameter $(0.1 + 0.2)^{\cancel{2}} \times 2 = 0.6$

$$P(Poi(0.6) = 0) = \frac{e^{-0.6}\, 0.6^0}{0!}$$

$$= e^{-0.6} \approx 0.5488.$$

~~19a THO~~

iv. The only paths under which the third jump is into C are BAC, CAC & CBC.

The probabilities of each jump is given by the ratio of the transition rates.

So the probabilities for each path are:

$$BAC = \frac{2}{3} \times \frac{1}{5} \times \frac{1}{3} = \frac{2}{45}$$

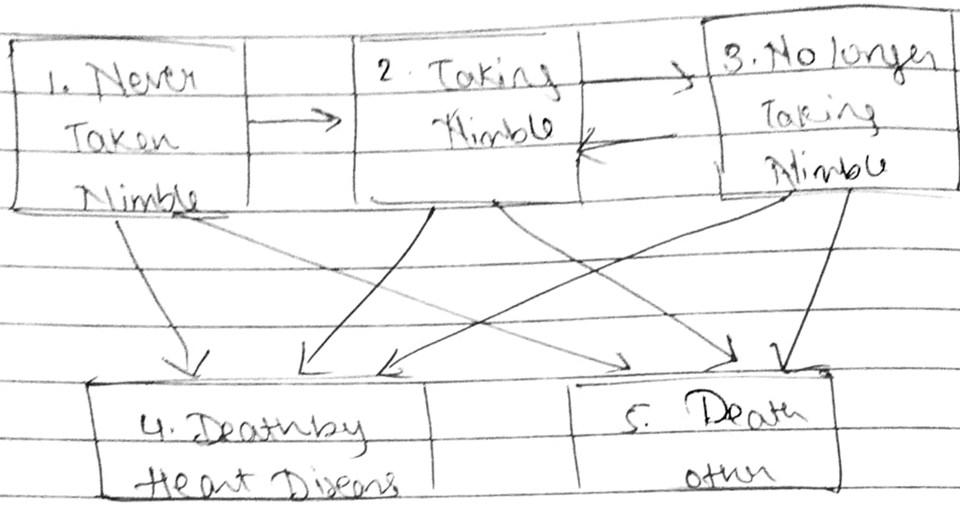
$$CAC = \frac{1}{3} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{12}$$

$$CBC = \frac{1}{3} \times \frac{1}{4} \times \frac{4}{5} = \frac{4}{60} = \frac{1}{15}$$

$$\text{Sum} = \frac{7}{36} = 0.194$$

Q3

i.



ii.

$${}_{dt+t}P_x^{34} = {}_tP_x^{31}\,{}_{dt}P_{x+t}^{14} + {}_tP_x^{32}\,{}_{dt}P_{x+t}^{24} + {}_tP_x^{33}\,{}_{dt}P_{x+t}^{34} + {}_tP_x^{34}\,{}_{dt}P_{x+t}^{44} + {}_tP_x^{35}\,{}_{dt}P_{x+t}^{54}$$

$$\because \ {}_{dt}P_{x+t}^{54} = {}_tP_x^{31} = 0$$

$${}_{dt+t}P_x^{34} = {}_tP_x^{32}\,{}_{dt}P_{x+t}^{24} + {}_tP_x^{33}\,{}_{dt}P_{x+t}^{34} + {}_tP_x^{34}\,{}_{dt}P_{x+t}^{44}$$

given that ${}_{dt}P_{x+t}^{44} = 1$

And assuming that, for small dt,

$${}_{dt}P_{x+t}^{ij} = \mu_{x+t}^{ij}\,dt + o(dt) \qquad i \neq j$$

where $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$

then substituting we have

$\cancel{{}_{dt}P^{ij}_{x+t}} =$

$${}_{dt+t}P^{34}_x = {}_tP^{32}_x \, \mu^{24}_{x+t}\, dt + P^{33}_x \, \mu^{34}_{x+t}\, dt + {}_tP^{34}_x + o(dt)$$

so that

$${}_{dt+t}P^{34}_x - {}_tP^{34}_x = {}_tP^{32}_x \, \mu^{24}_{x+t}\, dt + {}_tP^{33}_x \, \mu^{34}_{x+t}\, dt + o(dt)$$

& hence $\dfrac{d}{dt}\left({}_tP^{34}_x\right) = \lim\limits_{dt \to 0} \dfrac{{}_{t+dt}P^{34}_x - {}_tP^{34}_x}{dt}$

$$= {}_tP^{32}_x \, \mu^{24}_{x+t} + {}_tP^{33}_x \, \mu^{34}_{x+1}$$

Q4.

i. The mean is equal to the parameter, so there are 3 calls per hour.

ii. The process is memoryless so the fact that Fred has not had call for 15 minutes is irrelevant. Expected time until next call is 20 minutes.

iii. This is the probability of zero calls in time 0.5 hour

using $P_j(t) = e^{-\lambda t}(\lambda t)^j / j!$

OR

Since $P_0(0.5) = \dfrac{e^{-1.5}(1.5)^0}{0!}$

$P_0(0.5) = e^{-1.5} = 0.2231$

iv. The expected time that Fred is on the phone is the expected number of calls times the expected length of call.

Per hour this is 3 calls × 7 minutes = 21 minutes.

So the probability that the phone is engaged is

$21/60 = 0.35.$

Q5.

1. The Chapman Kolmogorov equation is

$$P_{HH}(x, t+dt) = P_{HH}(x,t)\,P_{HH}(t, t+dt) + P_{HS}(x,t)\,P_{SH}(t, t+dt) + P_{HD}(x,t)\,P_{HD}(t, t+dt)$$

But $P_{DH}(t, t+dt) = 0$ or other explanation why path through D can be ignored.

So :

$$P_{HH}(x, t+dt) = P_{HH}(x,t)\,P_{HH}(t, t+dt) + P_{HS}(x,t)\,P_{SH}(t, t+dt)$$

Assuming that for all small dt.

$$P_{ij}(t, t+dt) = \lambda_{ij}(t)\,dt + o(dt) \qquad i \neq j$$

$$P_{ij}(t, t+dt) = 1 + \lambda_{ii}(t)\,dt + o(dt)$$

~~OR~~

where ~~to~~ the λs are the instantaneous transition rates and $\lim_{dt \to 0} \frac{o(dt)}{dt} = 0$

then substituting, we have

$$P_{HH}(x, t+dt) = P_{HH}(x,t)\left(1 - \sigma(t)\,dt - \mu(t)\,dt\right) + P_{HS}(x,t)\,\rho(t, C_i) + O(dt)$$

so that

$$P_{HH}(x, t+dt) - P_{HH}(x,t) = P_{HH}(x,t)(-\sigma(t) - \mu(t))\,dt + P_{HS}(x,t)\,p(t, C_t)\,dt + o(dt)$$

& hence

$$\frac{d P_{HH}(x,t)}{dt} = \lim_{dt \to 0} \frac{P_{HH}(x, t+dt) - P_{HH}(x,t)}{dt}$$

$$= P_{HH}(x,t)\cancel{+dt}(-\sigma(t) - \mu(t)) + P_{HS}(x,t)\,p(t, C_t)$$

ii. The equation simplifies when considering $P_{\overline{HH}}(t)$

$$\frac{d}{dt} P_{\overline{HH}}(0,t) = -(\sigma(t) + \mu(t)) P_{\overline{HH}}(t)$$

$$\frac{1}{P_{HH}(0,t)} \frac{d}{dt} P_{\overline{HH}}(0,t) = -(\sigma(t) + \mu(t))$$

$$= \frac{d}{dt} \ln P_{\overline{HH}}(t).$$

Integrating both sides:

$$\left[\ln P_{\overline{HH}}(0,t)\right]_0^t = \int_{s=0}^{t} -(\sigma(s) + \mu(s))\,ds.$$

as $P_{\overline{HH}}(0) = 1$

$$P_{\overline{HH}}(0,t) = \exp - \left(\int_{s=0}^{t} (\sigma(s) + \mu(s))\,ds.\right.$$

Q6.

All three processors have a discreate state space.

A Markov Chain & Markov Jump Chain both operate in discrete time but a Markov Jump Process operates in continuous time.

All have the Markov property which is EITHER that the future development of the process can be predicted from its present state alone, without reference to its past history.

~~$P[X_t \in A \} | X_{s_t} = x_1, \}$~~

$$P[X_t \in A \mid X_{s_1} = x_1, X_{s_2} = x_2, \ldots, X_{s_n} = x_n, X_s = x] = P[X_t \in A \mid X_s = x]$$

for all times $s_1 < s_2 < \ldots < s_n < s < t$, all states $x_1, x_2, \ldots, x_n, x$ in S and all subsets A of S.

The Jump chain obeys the Markov Property & behave as a Markov Chain except when the Jump Chain encounters an absorbing state. From that time, it makes no further transitions, implying that time stops for the Jump Chain.

The Jump chain associated with X takes the same path through the state space as X does.

The Markov Jump chain and the Markov Chain are expressed in terms of probabilities whereas the Markov Jump Process is expressed in terms of rat

The Markov Chain can have loops in each state, the Markov Jump process cannot & the Markov Jump Chain only has loops on absorbing states.

Q7.

i. The maximum likelihood estimate of the transition intensity from state i to state j is the number of transitions from state i to state j divided by the total waiting time in state i.

To estimate the transition intensities exactly we therefore need the total time spent in each state.

OR

entry and exit times for each individual for each state & the total number of transitions of each type made.

ii. Define $P_{AA}(s,t)$ to be the probability of being in state Active at time $s+t$ if Active at time s.

Then EITHER

$$\frac{\partial}{dt} P_{AA}(s,t) = -P_{AA}(s,t)\mu$$

$$\frac{\partial}{\partial t} P_{AT}(s,t) = P_{AA}(s,t)\mu$$

OR

Integrated forward equations:

$$P_{AA}(s,t) = \exp\left(-\int_{u=s}^{t} \mu \, du\right)$$

$$P_{AT}(s,t) = \int_{u=0}^{t} P_{AA}(s,u) \cdot \mu \cdot 1 \, du$$

iii. Measure from time zero i.e. $s = 0$ & drop s from notation.

$$\frac{1}{P_{AA}(t)} \frac{d}{dt} P_{AA}(t) = -\mu$$

$$\frac{d}{dt}\left(\ln\left(P_{AA}(t)\right)\right) = -\mu$$

hence, $P_{AA}(t) = \exp(-\mu t + c)$

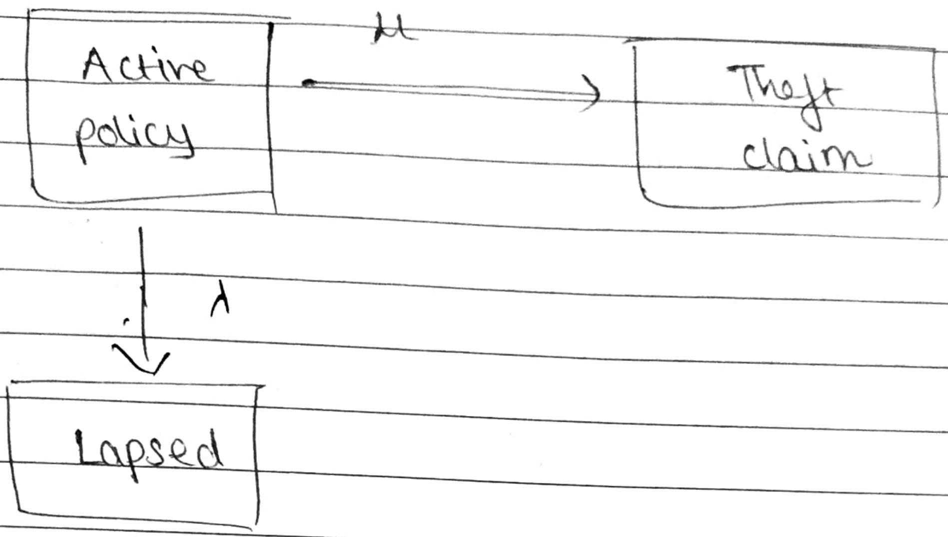
As $P_{AA}(0) = 1$, $c = 0$ so

$P_{AA}(t) = \exp(-\mu t)$

A claim occurs with cost £C if moves to the state "Theft Claim"

thence the expected cost is $C(1 - \exp(-\mu T))$

iv.



We now have $$\frac{d}{dt} P_{AA}(t) = -P_{AA}(t)(\mu+\lambda)$$

So $$P_{AA}(t) = \exp(-(\mu+\lambda)t)$$

We want $$\frac{d}{dt} P_{AT}(t) = P_{AA}(t)\mu = \mu \exp(-(\mu+\lambda)t)$$

Solving this produces.

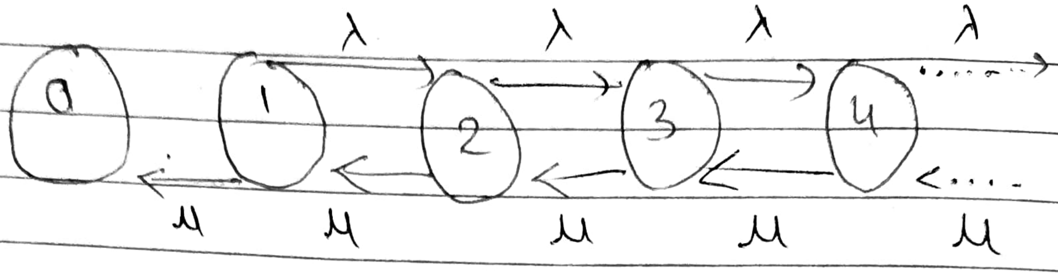
$$P_{AT}(t) = \left[\frac{-\mu}{(\mu+\lambda)} \exp(-(\mu+\lambda)t) \right]_0^T = \frac{\mu}{\mu+\lambda}\left(1-\exp(-(\mu+\lambda)T)\right)$$

So claim becoms $$\frac{\mu}{\mu+\lambda} C\left(1-\exp(-(\mu+\lambda)T)\right).$$

Q8.

i. $\{0, 1, 2, 3, 4 \ldots\}$

ii.



iii. Generator matrix.

$$\begin{array}{c|ccccc}
\text{Lives} & 0 & 1 & 2 & 3 & 4 \;\ldots \\
\hline
 & 0 & 0 & 0 & 0 & 0 \\
 & \mu & -(\mu+\lambda) & \lambda & 0 & 0 \\
 & 0 & 0 & -(\mu+\lambda) & \lambda & 0 \\
 & 0 & 0 & 0 & (-\mu+\lambda) & \lambda \\
 & 0 & 0 & 0 & 0 & (-\mu+\lambda)
\end{array}$$

iv. EITHER

If a Markov Jump process X_t is examined only at the times of transition, the resulting process is called the jump chain associated with X_t.

OR

A jump chain is each distinct state visited in the order visited where the time set is the times when states are moved between.

v.

$$
\begin{array}{cccccc}
\text{Lives} & 0 & 1 & 2 & 3 & 4 \; \ldots \\
 & \left[\begin{array}{c} 1 \\ \mu/(\mu+\lambda) \\ 0 \\ 0 \\ 0 \end{array}\right. & \begin{array}{c} 0 \\ 0 \\ \mu/(\mu+\lambda) \\ 0 \\ 0 \end{array} & \begin{array}{c} 0 \\ \lambda/(\mu+\lambda) \\ 0 \\ \mu/(\mu+\lambda) \\ 0 \end{array} & \begin{array}{c} 0 \\ 0 \\ \lambda/(\mu+\lambda) \\ 0 \\ \mu/(\mu+\lambda) \end{array} & \left.\begin{array}{c} 0 \\ 0 \\ 0 \\ \lambda/(\mu+\lambda) \\ 0 \end{array}\right] \text{etc.}
\end{array}
$$

etc

vi. $\left(\dfrac{\mu}{\mu+\lambda}\right)^3$

Q9.

i. The transition rates of moving from each state to each other state must be non-negative.
OR The transition rats OFF the leading diagonal must be must be non-negative
OR $\mu_{ij} >= 0$ for $i \neq j$

The transition rates ON the leading diagonal must be non-positive.
OR $\mu_{ij} <= 0$

The sum of each row must be zero.
OR $\mu_{ij} = \cancel{\Sigma_{j \neq \dots}} - \Sigma_{j \neq i} \mu_{ij}$

Should be finite or countable square matrix.

ii. State space {2, 1, 0} policies in force.

iii.

$$2 \xrightarrow{0.4} 1 \xrightarrow{0.2} 0$$

iv. $\frac{d}{dt} P_{22}(t) = -0.4 P_{22}(t)$

$\frac{d}{dt} P_{21}(t) = 0.4 P_{22}(t) - 0.2 P_{21}(t)$

$\frac{d}{dt} P_{20}(t) = 0.2 P_{21}(t)$

Q10.

i. $\frac{d}{dt} P_{\bar{A}\bar{A}}(t) = -2t \times P_{\bar{A}\bar{A}}(t)$

$\Rightarrow \frac{d}{dt}\left[\ln P_{\bar{A}\bar{A}}(t)\right] = -2t$

$\Rightarrow \ln P_{\bar{A}\bar{A}}(s) = -s^2 + \text{constant}$

We know $P_{\bar{A}\bar{A}}(0) = 1$, hence constant $= 0$

Hence, $P_{\bar{A}\bar{A}}(s) = \exp^{-s^2}$

ii. P(in first visit to B at time T | in state A at $t = 0$)

$= \int_0^T P$(remains in A to time s)

$\times$ P(transition to B in time s, s+ds)

$\times$ P(remains in B to time T) ds

$$= \int_{s=0}^{T} P_{\bar{A}\bar{A}}(s) \times 2s \times P_{\bar{B}\bar{B}}(s, T)\, ds$$

Using the result from part (i) & the similar result for P_{BB} with boundary condition $P_{BB}(s,s) = 1$, this gives us:

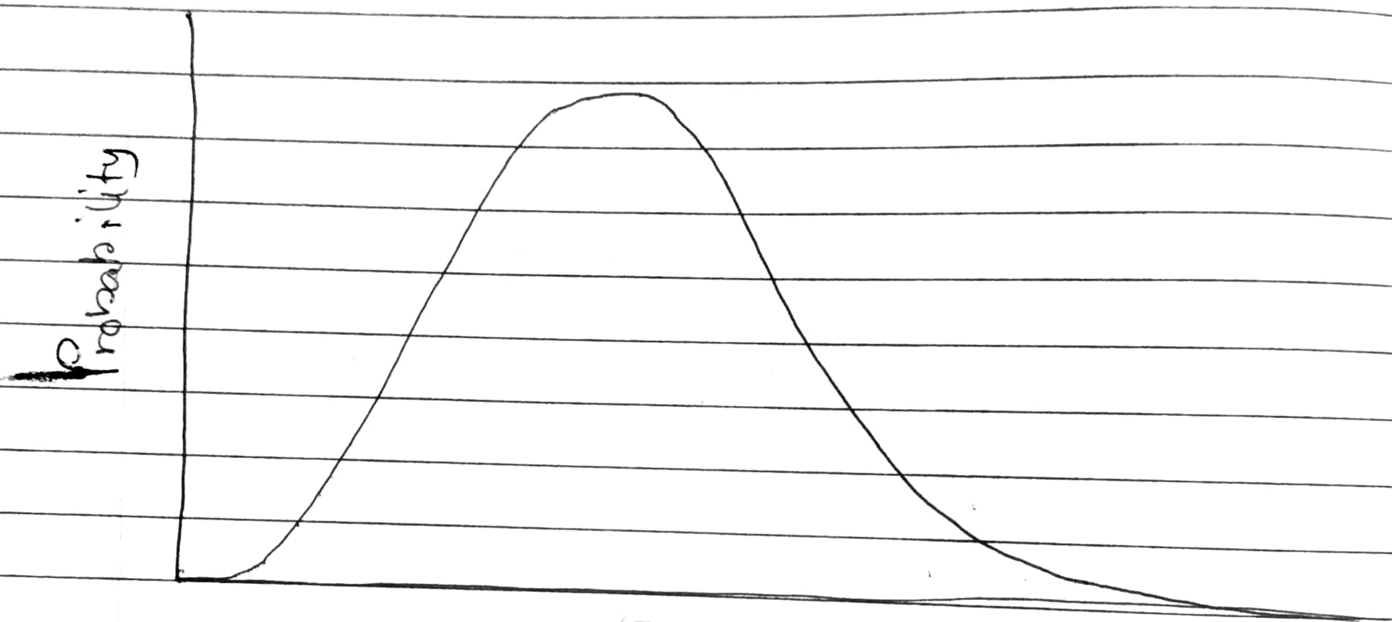
$$= \int_{s=0}^{T} e^{-s^2} \times 2s \times e^{-T^2+s^2}\, ds$$

$$= \int_{s=0}^{T} 2s \times e^{-T^2}\, ds$$

$$= e^{-T^2} \times T^2$$

iii.

(a) The sketch should be shaped like :



(b) Commentary :

- Initially probability increases from 0 at $T = 0$ and accelerates as the trasition rate from A to B res.

- However, as transitions increase, it becomes more likely that the process has already visited state B and jumped back to A.
  Therefore the probability of being in the first visit to B tends (exponentially) to zero.

(c) Differentiate to find turning point :

$$\frac{d}{dt}\left[e^{-t^2} \times t^2\right] = 2t \times e^{-t^2} - 2t^3 \times e^{-t^2}$$

set derivative equal to zero,

$$e^{-t^2} \times 2t \times (1-t^2) = 0$$

implies $t=1$ for a positive solution and, from above analysis, this is clearly a maximum.

Q11.

i. Let N_t denote the number of claims up to time t. Since the Poisson Process has stationary increments, we may take $t=0$, so then the required conditional distribution is

$$P(T_0 \le y \mid N_s = 1) = \frac{P(T_0 \le y, N_s = 1)}{P(N_s = 1)}$$

$$= \frac{P(N_y = 1, N_s - N_y = 0)}{P(N_s = 1)}$$

But $N_s - N_y$ is independant of N_y & has the same distibution as N_{s-y}

Thus the right hand side above equals

$$\frac{(\lambda y e^{-\lambda y})\, e^{-\lambda(s-y)}}{\lambda s e^{-\lambda s}} = \frac{y}{s}$$

which is the cdf of the uniform distribution on $[0,s]$

ii. Since holding times are independent, each having an exponential distribution, their joint density is

$$\lambda^n e^{-\lambda(t_1 + t_2 + \cdots + t_n)} \quad 1_{\{t_1, t_2, \ldots, t_n > 0\}}$$

iii. We have, as in part (i),

$$P(N_s = k \mid N_t = n) = \frac{P(N_s = k,\ N_t = n)}{P(N_t = n)}$$

$$= \frac{P(N_s = k,\ N_t - N_s = n-k)}{P(N_t = n)}$$

Using again that the poisson process has stationary & independant increments & that the number of claims in an interval $[0, t]$ is Poisson (λt), we derive from above that

$$P(N_s = k \mid N_t = n) = \frac{\dfrac{e^{-\lambda s}(\lambda s)^k}{k!} \times \dfrac{e^{-\lambda(t-s)}\lambda^{n-k}(t-s)^{n-k}}{(n-k)!}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{e^{-\lambda t}\lambda^n s^k (t-s)^{n-k}}{k!\,(n-k)!} \times \frac{n!}{e^{-\lambda t}\lambda^n t^n}$$

$$= \frac{n!}{k!\,(n-k)!} \times \frac{s^k (t-s)^{n-k}}{t^k\, t^{n-k}}$$

$$= \binom{n}{k}\left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}$$

which is binomial with parameters n & s/t.