Subject : Probability and statistics - 1
Chapter : Unit 3 and 4
Assignment 2

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Answer 1 -

Let X be the claim size for the home insurance policy
$X \sim N(800, 100^2)$
Let Y be the claim size for the car insurance policy
$Y \sim N(1200, 300^2)$

$$Y_1 + Y_2 + Y_3 - (X_1 + X_2 + X_3 + X_4) \sim N(3(1200) - 4(800), 3(300)^2 + 4(100)^2)$$

The requirement follows $N(400, 310000)$
Standardising Normal Distribution $\longrightarrow$

$$P\left(Z > \frac{800-400}{\sqrt{310000}}\right) = P(Z > 0.718) = 1 - P(Z < 0.718) = 0.23576$$

Answer 2 -

$$P(N=n) = \binom{n+2}{n} 0.9^3\, 0.1^n = {}^{n+2}C_n\, 0.9^3\, 0.1^n$$

$N \sim$ Neg. Binomial$(3, 0.9)$

Let the claim amount be X (in 000's)
Then $X \sim$ Gamma$(6, 2)$
Also, Y is the total claim amount

$$M_Y(t) = M_n \log(M_X(t))$$

$$\because M_X(t) = \left(1 - \frac{t}{2}\right)^{-6}$$

$$M_Y(t) = M_N \log\left(1 - \frac{t}{2}\right)^{-6}$$

$$= \left[\frac{pe^{\log(1-t/2)^{-6}}}{1 - qe^{\log(1-t/2)^{-6}}}\right]^3 \qquad \because M_n(t) = \left(\frac{pe^t}{1-qe^t}\right)^k$$

$$= \left[\frac{0.9(1-t/2)^{-6}}{1-(0.1)(1-t/2)^{6}}\right]^3$$

$$E(N) = \frac{0.3}{0.9} = 0.333$$

$$V(N) = \frac{k(0.1)}{(0.9)^2} = 0.37$$

$E(X) = \frac{\alpha}{\lambda} = 3$ $\qquad V(X) = \frac{\alpha}{\lambda^2} = \frac{6}{4} = 1.5$

$V(Y) = E(N)\,V(X) + E(X)^2\,V(N)$

$= 5 + \frac{10}{3} = 8.333$

$SD(Y) = \sqrt{V(X)} = \sqrt{8.333} = 2.88$

Answer 3 —

$f_x(x) = \lambda e^{-\lambda(x-\alpha)}$

$M_x(t) = E(e^{tx})$

$= \int_{\alpha}^{\infty} e^{tx}\,\lambda e^{-\lambda(x-\alpha)}\,dx$

$= \lambda e^{\lambda\alpha} \int_{\alpha}^{\infty} e^{x(t-\lambda)}\,dx$

$= \frac{\lambda e^{\lambda\alpha}}{t-\lambda}\left[e^{(t-x)x}\right]_0^{\infty} =$

$= \frac{\lambda e^{\lambda\alpha}}{t-\lambda}\left(0 - e^{-\lambda\alpha}e^{t\alpha}\right)$

$M_x(t) = \frac{\lambda e^{t\alpha}}{\lambda - t}$

(ii) $M'_x(t) = \lambda(e^{t\alpha})(\lambda-t)^{-2}(-1)(-1) + \lambda(\lambda-t)^{-1}(e^{t\alpha})(\alpha)$

$= \frac{\lambda e^{t\alpha}}{(\lambda-t)^2} + \frac{\alpha\lambda e^{t\alpha}}{(\lambda-t)}$

$E(X) = M'_x(0) = \frac{\lambda}{\lambda^2} + \frac{\lambda\alpha}{\lambda} = \frac{1+\lambda\alpha}{\lambda}$

$M''_x(t) = \lambda e^{t\alpha}(-2)(\lambda-t)(-3)(-1) + \lambda(\lambda-t)^{-2}e^{t\alpha}(\alpha) + \lambda\alpha e^{t\alpha}(\lambda-t)^{-2}$

$+ \lambda\alpha(\lambda-t)^{-1}e^{t\alpha}(\alpha)$

$M''_x(t) = \frac{2\lambda e^{t\alpha}}{(\lambda-t)^3} + \frac{\alpha\lambda e^{t\alpha}}{(\lambda-t)^2} + \frac{\lambda\alpha e^{tx}}{(\lambda-t)^2} + \frac{\alpha^2\lambda e^{t\alpha}}{(\lambda-t)}$

$E(X^2) = M''_x(0) = \frac{2}{\lambda} + \frac{2\alpha}{\lambda} + \alpha^2$

$V(X) = E(X^2) - (E(X))^2 = \alpha^2 + \frac{2\alpha}{\lambda} + \frac{2}{\lambda^2} - \left(\frac{1+\lambda^2\alpha^2+2\alpha\lambda}{\lambda^2}\right)$

$= \alpha^2 + \frac{2\alpha}{\lambda} + \frac{2}{\lambda^2} - \frac{1}{\lambda^2} - \alpha^2 - \frac{2\alpha}{\lambda} = \frac{1}{\lambda^2}$

Answer 4 -

$$G_v(t) = \left(\frac{p}{1-qt}\right)^k \qquad p+q=1$$

Resembles negative binomial distribution

$$P(V=4) = \frac{\sqrt{k+4}}{\sqrt{5}\sqrt{k}} \, p^4 q^4$$

Answer 5 -

$$f(x,y) = ax^2 + axy \qquad 0<x<5,\ 10<y<20$$

$$\int f(x,y) = \int_{10}^{20}\int_0^5 ax^2 + axy \, dx\, dy$$

$$1 = \int_{10}^{20} \frac{125a}{3} + \frac{25}{2}ay \, dy$$

$$1 = \frac{125a}{3}\left(y\right)_{10}^{20} + \frac{25}{2}a\left(\frac{y^2}{a}\right)_{10}^{20}$$

$$\therefore a = \frac{3}{6875}$$

$$f_x(x) = a\int_{10}^{20} x^2 + xy \, dy$$

$$= \frac{3}{6875}\int_{10}^{20} x^2 + xy \, dy$$

$$= \frac{3}{6875}\left(10x^2 + 150x\right)$$

$$E(Y/X) = y f(Y/X) = y\int_{10}^{20} \frac{f(x,y)}{f_x(x)} = \int_{10}^{20} \frac{y\left(\frac{3}{6875}\right)\left(x^2+xy\right)dy}{\frac{3}{6875}\left(10x^2+150x\right)}$$

$$E\left(\frac{Y}{X}\right) = \frac{45x + 7000}{3(x+15)}$$

$$P\left(Y > \frac{1}{3}x + 10\right)$$

$$f_Y(y) = \int_0^5 \frac{3}{6875}\left(x^2+xy\right)dx = \frac{1}{6875}\left(\frac{250 + 75y}{2}\right)$$

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$$P\left(Y > \frac{1}{3}X + 10\right) = \int_{\frac{x}{3}+10}^{20} \frac{1}{6875}\left(\frac{250+75y}{2}\right) dy$$

$$= \frac{1}{6875(2)}\left[10000 - \frac{1750x}{3} - \frac{25x^2}{3}\right].$$

Answer 6 – $X \sim Poi(\lambda)$ $\quad Y \sim Poi(\mu)$

$$M_X(t) = e^{\lambda(e^t-1)} \qquad M_Y(t) = e^{\mu(e^t-1)}$$

$$M_{X-Y}(t) = E\left(e^{t(x-y)}\right)$$
$$= E\left(e^{tX-\cancel{ty}}\right)$$
$$= E\left(\frac{e^{tx}}{e^{ty}}\right) = \boxed{e^{\lambda-\mu\,(e^t-1)}}$$

$$M_{X+Y}(t) = E\left(e^{t(x+y)}\right)$$
$$= E(e^{tx})\,E(e^{ty})$$
$$= e^{\lambda(e^t-1)}\,e^{\mu(e^t-1)}$$
$$= \boxed{e^{\lambda+\mu\,(e^t-1)}}$$

Answer 7 – $V\left(\frac{X}{Y}=2\right) = E\left(\frac{X^2}{Y}=2\right) - \left(E\left(\frac{X}{Y}=2\right)\right)^2$

$$= \sum_x \frac{x^2 P(X=x, Y=y)}{P(Y=2)} - \left(\sum_x x\,\frac{P(X=x, Y=y)}{P(Y=y)}\right)^2$$

$$= \frac{1^2(0) + 2^2(0.3) + 3^2(0.1) + 4^2(0.1)}{0.6} - \left[1(0) + 2(0.3) + 3(0.1) + 4(0.1)\right]$$

$$= \frac{53}{6} - \frac{289}{30} = \frac{39}{36}$$

(ii) $f(u,v) = \frac{48}{67}(2uv - v^2) \qquad 0 < u < 1 \quad \frac{u}{2} < v < 2$

$$E\left(\frac{U}{V}=v\right) = \int_0^1 u\, f\left(\frac{U}{V}=v\right) du$$

$$= \int_0^1 \frac{u f(v, v=0)}{f(V=v)}\, v \quad \text{———} \; ①$$

$$f_v(v=u) = \int_0^1 \frac{48}{67}\left(2uv - v^2\right) du$$

$$= \frac{48}{67}\left(\frac{2u^2v}{2} - \frac{v^3}{3}\right)_0^1$$

$$= \frac{16}{67}(3v-1)$$

$$E(U/V=v) = \frac{\int_0^1 v\left(\frac{48}{67}\right)\left(2uv - u^2\right) du}{\frac{16}{67}(3v-1)}$$

$$= \frac{3}{3v-1}\int_0^1 2u^2v - v^3\, du = \frac{3}{3v-1}\left[\frac{3v}{3} - \frac{1}{u}\right]$$

$$= \frac{8v-3}{4(3v-1)}$$

Answer 8 –

$\alpha = 3, \ \lambda = 2$

$X \sim Gamma(3,2)$

$E(Y/X=x) = 3x+1$

$V(Y/X=x) = 2x^2+5$

$$E(Y) = E\left(E(Y/X=x)\right)$$
$$= E(3x+1)$$
$$= 3E(X)+1$$
$$= 3(3/2)+1 = 11/2$$

$$V(Y) = E\left(V(Y/X=x)\right) - V\left(E(Y/X=x)\right)$$
$$= E(2x^2+5) + V(3x+1)$$
$$= 2(3)+5+9(3/4) = 17.75$$

Answer 9 –

$Z = X+Y$

$E(X) = 3 \qquad \sigma_X = \sqrt{V(X)} = 2$

$E(Y) = 4 \qquad \sigma_Y = \sqrt{V(Y)} = 1$

$$cov(X,Z) = cov(X, X+Y) = cov(X,X) + cov(X,Y)$$
$$= V(X) - 0.6 = 3.4$$

$V(Z) = var(X+Y)$

$= V(X) + (V(Y)) + 2cov(X+Y)$

$= 4 + 1 + 2(0.06)$

$= 3.8$

Answer 10 –

$f(x,y) = kx^{-\alpha} e^{-y/\beta}$ $\quad 1 < x < \infty$

$1 < y < \infty$

$\alpha > 1 \quad \beta > 0$

$1 = k \int_1^\infty \int_1^\infty x^{-\alpha} e^{-y/\beta} \, dx \, dy$

$1 = k \left(\frac{1}{-\alpha+1} \right) \left(-\beta e^{-1/\beta} \right)$

$\boxed{k = \frac{\alpha - 1}{\beta e^{-1/\beta}}}$