Probability & statistics Assignment II

1) X = Claim size on a home insurance

$X \sim \text{Normal}(800, 100^2)$

Y = Claim size on a car insurance

$Y \sim \text{Normal}(1200, 300^2)$

$P(Y_1 + Y_2 + Y_3 > X_1 + X_2 + X_3 + X_4 + 800)$

$= P((Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) > 800)$

$(Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) \sim N(3(1200) - 4(800), (3 \times 300^2 + 4 \times 100^2))$

$\sim N(400, 310000)$

$P((Y_1 + Y_2 + Y_3) - (X_1 + X_2 + X_3 + X_4) > 800)$

$= P\left(Z > \frac{800 - 400}{\sqrt{310000}}\right) = P(Z > 0.718)$

$= 1 - P(Z \leq 0.718)$

$= 1 - 0.76424$

$= 0.23576$

2) N = number of claims

$N \sim$ Negative Binomial $(3, 0.4)$

X = Claim amount

$X \sim \text{Gamma}(6, 2)$

Y = Total Claim Amount

$$M_y(t) = N_N(\log M_x(t))$$

$$M_x(t) = \left(1 - \frac{t}{2}\right)^{-6}$$

$$M_y(t) = M_N\left(\log\left(1 - \frac{t}{2}\right)^{-6}\right)$$

$$M_x(t) = \left(1 - \frac{t}{2}\right)^{-6}$$

$$M_y(t) = M_N\left(\log\left(1 - \frac{t}{2}\right)^{-6}\right)$$

$$\ldots \left(M_N(t) = \left(\frac{pe^t}{1-qe^t}\right)^k\right)$$

$$M_y(t) = \left(\frac{pe^{\log\left(1-\frac{t}{2}\right)^{-6}}}{1-qe^{\log\left(1-\frac{t}{2}\right)^{-6}}}\right)^3$$

$$M_y(t) = \left(\frac{0.9\left(1-\frac{t}{2}\right)^{-6}}{1-(0.1)\left(1-\frac{t}{2}\right)^{-6}}\right)^3$$

$$E(N) = \frac{3}{0.9} = \frac{10}{3}$$

$$V(N) = \frac{k(0.1)}{(0.9)^2} = \frac{3}{0.9}\left(\frac{1}{9}\right) = \frac{10}{27}$$

$$E(X) = \frac{\alpha}{\lambda} = \frac{6}{2} = 3 \qquad V(X) = \frac{\alpha}{\lambda^2} = \frac{6}{4} = \frac{3}{2}$$

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$$V(Y) = E(N)\,V(X) + E(X)^2\,V(N)$$

$$= \frac{10}{3}\left(\frac{3}{2}\right) + 9\left(\frac{10}{3}\right)$$

$$= 5 + \frac{10}{3} = \frac{25}{3}$$

$$S.D(Y) = \sqrt{\frac{25}{3}} \approx 2.886$$

3) $f(x) = \lambda e^{-\lambda(x-\alpha)}$

$$M_X(t) = E(e^{tx})$$

$$= \int_{\alpha}^{\infty} e^{tx}\,\lambda e^{-\lambda(x-\alpha)}\,dx$$

$$= \int_{\alpha}^{\infty} e^{tx}\, e^{-\lambda x}\, e^{\lambda\alpha}\,dx$$

$$= \lambda e^{\lambda\alpha}\int_{\alpha}^{\infty} e^{x(t-\lambda)}\,dx$$

$$= \frac{\lambda e^{\lambda\alpha}}{t-\lambda}\left[e^{(t-\lambda)x}\right]_{0}^{\infty} = \frac{\lambda e^{\lambda\alpha}}{t-\lambda}\left[e^{-(\lambda-t)x}\right]_{0}^{\infty}$$

$$M_X(t) = \frac{\lambda e^{\lambda\alpha}}{t-\lambda}\left[0 - e^{-\lambda\alpha}\cdot e^{t\alpha}\right]$$

$$M_X(t) = \frac{\lambda e^{t\alpha}}{\lambda - t}$$

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u) $M_x'(t) = \lambda(e^{t\alpha})\left(\lambda-t\right)^{-2}(-1)(-1) + \lambda(\lambda-t)^{-1}(e^{t\alpha})(\alpha)$

$$= \frac{\lambda e^{t\alpha}}{(\lambda-t)^{+2}} + \frac{\lambda e^{t\alpha}\alpha}{\lambda - t}$$

$$E(x) = M_x'(0) = \frac{\lambda(1)}{\lambda^2} + \frac{\lambda\alpha}{\lambda}$$

$$= \frac{1+\lambda\alpha}{\lambda}$$

$$= \frac{1+\lambda\alpha}{\lambda}$$

$$M''_x(t) = \lambda e^{t\alpha}(-2)(\lambda-t)^{-3}(-1) + \lambda(\lambda-t)^{-2}(e^{t\alpha})(\alpha).$$
$$+\lambda\alpha e^{t\alpha}(\lambda-t)^{-2}(-1)(-1) + \lambda\alpha(\lambda-t)^{-1}(e^{t\alpha})(\alpha)$$

$$= \frac{2\lambda e^{t\alpha}}{(\lambda-t)^3} + \frac{\lambda\alpha e^{t\alpha}}{(\lambda-t)^2} + \frac{\lambda\alpha e^{t\alpha}}{(\lambda-t)^2} + \frac{\alpha^2\lambda e^{t\alpha}}{\lambda-t}$$

$$= \frac{2\lambda e^{t\alpha}}{(\lambda-t)^3} + \frac{2\lambda\alpha e^{t\alpha}}{(\lambda-t)^2} + \frac{\lambda\alpha^2 e^{t\alpha}}{\lambda-t}$$

$$E(x^2) = M''_x(0) = \frac{2\lambda(1)}{\lambda^2} + \frac{2\lambda\alpha}{\lambda^2} + \frac{\lambda\alpha^2}{\lambda}$$

$$= \frac{2}{\lambda^2} + \frac{2\alpha}{\lambda} + \alpha^2$$

$$V(x) = E(x^2) - (E(x))^2$$
$$= \alpha^2 + \frac{2\alpha}{\lambda} + \frac{2}{\lambda^2} - \left(\frac{1+\lambda^2\alpha^2+2\alpha\lambda}{\lambda^2}\right)$$

$$V(x) = \frac{1}{\lambda^2}$$

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4) $G_V(t) = \left(\dfrac{p}{1-q^t}\right)^k \quad , \quad p+q=1$

$\therefore$ Given distribution is negative binomial type 2

$P(V=4) \quad \cancel{\dfrac{\sqrt{k+4}}{\quad}} \quad \dfrac{\sqrt{k+4}}{\sqrt{5}\sqrt{k}} \cdot p^4 q^4$

5) $f(x,y) = ax^2 + axy \qquad \begin{array}{l} 0 < x < 5 \\ 10 < y < 20 \end{array}$

$$1 = \int_{10}^{20}\int_0^5 ax^2 + axy \; dx \, dy$$

$$1 = \int_{10}^{20} \left[\frac{ax^3}{3}\right]_0^5 + \left[ay\frac{x^2}{2}\right]_0^5 dy$$

$$1 = \int_{10}^{20} \frac{125a}{3} + \frac{25}{2}ay \; dy$$

$$1 = \frac{125a}{3}(y)_{10}^{20} + \frac{25}{2}a\left[\frac{y^2}{2}\right]_{10}^{20}$$

$$1 = \frac{1250a}{3} + 1875a$$

$$a = \frac{3}{6875}$$

$$f_x(x) = \int_{10}^{20} (x^2 + xy)\,a\,dy$$

$$= \frac{3}{6875} \int_{10}^{20} (x^2 + xy)\,dy$$

$$= \frac{3}{6875} \left(10x^2 + 150x\right)$$

$$E(V/X) = y\,f(V/x) = y \int_{10}^{20} \frac{f(x,y)}{f_x(x)}$$

$$= \frac{1}{10} \int_{10}^{20} \frac{y(x+y)}{10(x+15)}\,dy$$

$$= \frac{1}{10(x+15)} \left[\left[x \cdot \frac{y^2}{2} + \frac{y^3}{3}\right]_{10}^{20}\right]$$

$$= \frac{1}{x+15} \left(15x + \frac{700}{3}\right)$$

$$= \frac{45x + 700}{3(x+15)}$$

$$E(V/x) = \frac{45x + 700}{3(x+15)}$$

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$$P\left(Y > \frac{1}{3}x + 10\right) \Rightarrow$$

$$f_Y(y) = \int_0^5 \frac{3}{6875} (x^2 + xy)\,dx$$

$$= \frac{3}{6875}\left[\frac{x^3}{3} + \frac{x^2}{2}y\right]_0^5$$

$$= \frac{3}{6875}\left[\frac{125}{3} + \frac{25y}{2}\right]$$

$$= \frac{1}{6875}\left[\frac{250 + 75y}{2}\right]$$

$$P\left(Y > \frac{1}{3}x + 10\right) = \int_{1/3 x + 10}^{20} \frac{1}{6875} \cdot \frac{(250 + 75y)}{2}\,dy$$

$$= \frac{1}{6875(2)}\left[250\left[y\right]_{1/3 x + 10}^{20} + 75\left[\frac{y^2}{2}\right]_{x/3 + 10}^{20}\right]$$

$$= \frac{1}{6875(2)}\left[5000 - \frac{250x}{3} - 2500 + 15000 - \frac{25x^2}{3} - 7500 - \frac{1500x}{3}\right]$$

$$= \frac{1}{6875(2)}\left[1000 - \frac{1750}{3}x - \frac{25}{3}x^2\right]$$

6) $X \sim Poi(\lambda) \quad Y \sim Poi(\mu)$

$M_x(t) = e^{\lambda(e^t - 1)} \qquad M_y(t) = e^{\mu(e^t - 1)}$

$$M_{x-y}(t) = E\left[e^{t(x-y)}\right]$$
$$= E\left(e^{xt - yt}\right)$$
$$= e^{(\lambda - \mu)(e^t - 1)}$$

$\therefore X - Y \sim Poi(\lambda - \mu)$

Hence proved

$$M_{x+y}(t) = E\left(e^{t(x+y)}\right) = E\left(e^{tx + yt}\right)$$
$$= e^{\lambda(e^t - 1)} \cdot e^{\mu(e^t - 1)}$$
$$= e^{(\lambda + \mu)(e^t - 1)}$$

$\therefore X + Y \sim Poi(\lambda + \mu)$

$\therefore$ Hence Proved

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7) $Var(X/Y=2) = E(X^2/Y=2) - (E(X/Y=2))^2$

$= \sum_x x^2 \frac{P(X=x, Y=2)}{P(Y=2)} - \left[\sum_x x \frac{P(X=x, Y=2)}{P(Y=2)}\right]^2$

$= \frac{1^2(0) + 2^2(0.3) + 3^2(0.1) + 4^2(0.2)}{0.6} - \left[\frac{1(0) + 2(.3) + 3(0.1) + 4(0.2)}{0.6}\right]^2$

$= \frac{53}{6} - \frac{289}{36}$

$= \frac{29}{36}$

11) $f(U,V) = \frac{48}{67}(2uv - v^2)$, $0 < v < 1$, $\frac{v}{2} < u < 2$

$E(U/V=v) = \int_0^1 u f(U/V=v) \, du$

$= \int_0^1 u \frac{f(u, V=v)}{f(V=v)} \, du$ - 1

$\Rightarrow f_v(V=v) = \int_0^1 \frac{48}{67}(2uv - v^2) \, du$

$= \frac{48}{67}\left[\frac{\cancel{2}u^2}{\cancel{2}} v - \frac{u^3}{3}\right]_0^1$

$= \frac{48}{67}\left[v - \frac{1}{3}\right]$

$= \frac{16}{67}(3v - 1)$

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$$E(U/V=v) = \int_0^1 \frac{u\frac{(48)}{67}(2uv - v^2)}{\frac{16}{67}(3v-1)}\, du$$

$$= \frac{3}{(3v-1)} \int_0^1 (2u^2v - v^3)\, du$$

$$E(U/V=v) = \frac{(8v-3)}{4(3v-1)}$$

8) $X \sim \text{Gamma}(3,2)$

$E(Y/X=x) = 3x+1$ $\qquad$ $Var(Y/X=x) = 2x^2+5$

$E(E(Y/X=x)) = E(Y)$

$E(Y) = E(3x+1)$

$= 3E(x)+1$

$= 3\left(\frac{3}{2}\right)+1$

$= \frac{11}{2}$

$Var(Var(Y/X=x)) = V(Y)$

$V(Y) = E(Var(Y/\cancel{X}=x))$

$+ V(E(Y/X=x))$

$= E(2x^2+5) + V(3x+1)$

$= 2E(x^2)+5 + 9v(x)$

$= 6+5+6.75$

$= 17.75$

9) $Z = X+Y$

$E(x) = 3$ $\qquad$ $\sigma = (4) = 4$

$\sqrt{v(x)} = 2$ $\qquad$ $\sqrt{v(y)} = 1$

i) $Cov(x,z) = Cov(x, x+y)$ $\qquad$ $Cov(x,y) = -0.3$

$= Cov(x,x) + Cov(x,y)$

$= V(x) + (-0.6)$

$= 3.4$

9) $Var(z) = Var(x+y)$

$= V(x) + V(x) + 2Cov(x,y)$

$= 4 + 1 + 2(-0.6)$

$= 5 - 1.2$

$= 3.8$

10) $f(x,y) = kx^{-\alpha} e^{-y/\beta}$, $1 < x < \infty$

$1 < y < \infty$

$\alpha > 1$, $\beta > 0$

$$1 = k\int_1^\infty x^{-\alpha}\,dx \int_1^\infty e^{-y/\beta}\,dy$$

$$1 = k\left[\frac{x^{-\alpha+1}}{-\alpha+1}\right]_1^\infty \left[\frac{e^{-y/\beta}}{-1/\beta}\right]_1^\infty$$

$$1 = k\left[0 - \frac{1}{-\alpha+1}\right]\left[0 + \beta\left(e^{-1/\beta}\right)\right]$$

$$\therefore K = \frac{-1+\alpha}{\beta e^{-1/\beta}} = \frac{\alpha-1}{\beta e^{-1/\beta}} = k$$