Probability & Statistics Assignment - 2

1. Let A be claim prize on home insurance and B be on car insurance

$A \sim N(800, 100^2)$ and $B \sim N(1200, 300^2)$

$P(B_1 + B_2 + B_3 > A_1 + A_2 + A_3 + A_4 + 800)$

$P(B_1 + B_2 + B_3 - (A_1 + A_2 + A_3 + A_4) > 800)$

$(B_1 + B_2 + B_3) - (A_1 + A_2 + A_3 + A_4) \sim N(3(1200) - 4(800))$

$\sim N(400, 310000)$

$P\left(C > \frac{800-400}{\sqrt{310000}}\right)$

$P(Z > 0.71842) \Rightarrow 1 - P(C < 0.71842)$

$= 1 - 0.76424$

$\Rightarrow \boxed{0.23576}$

2. $N \sim$ Negative Binomial (k, P)

$P(N = n) \Rightarrow {}^{n+2}C_n (0.9)^3 (0.1)^n$

$\Rightarrow {}^{(n+2)-1}C_{n-1} (0.9)^3 (0.1)^n$

where $k = 3$, $p = 0.9$.

$X \sim$ Gamma $(6, 2)$

Y is total amount of claim

$M_Y t = M_N(\log M_X t)$

(i) MGF of Y $\Rightarrow$ $M_Y(t) = E(e^{yt}/N=n)$

$E(e^{yt}/N=n) = E(e^{x_1 t + x_2 t + x_3 t \dots x_n t})$

$= E(e^{tx_1}) \cdot E(e^{tx_2}) \dots E(e^{tx_n})$

$= \left(1 - \frac{t}{2}\right)^{-6n}$

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$$E\left(E\left(e^{Yt}(N=n)\right)\right) \Rightarrow E\left(\left(1-\frac{t}{2}\right)^{-6n}\right)$$

$$\Rightarrow E\left(e^{N\log\left(1-\frac{t}{2}\right)^{-6}}\right)$$

$$\Rightarrow M_N\left(\log\left(1-\frac{t}{2}\right)^{-6}\right).$$

3. $f(x) = \lambda e^{-\lambda(x-a)}$

$MGF \Rightarrow E(e^{tx}) \Rightarrow \int_a^\infty e^{tx} \lambda e^{-\lambda(x-a)} \, dx.$

$M_x t = \lambda \int_a^\infty e^{tx} \cdot e^{-\lambda x} \cdot e^{\lambda a} \, dx \Rightarrow e^{\lambda \alpha} \lambda \int_a^\infty e^{-(\lambda - t)x} \, dx$

$= \frac{e^{a\lambda} \lambda}{t-\lambda} \left[e^{-(\lambda-t)x} \right]_a^\infty \cancel{\Rightarrow e^{a\lambda} \int_a^\infty e^{-(\lambda-t)x} \, dx}$

$\Rightarrow \frac{e^{a\lambda}}{t-\lambda} \lambda \left[-e^{-(\lambda-t)a} \right]_a^\infty$

$\Rightarrow \frac{\lambda e^{\lambda a}}{t-\lambda} \left[0 - e^{-\lambda \alpha} \cdot e^{t\alpha} \right.$

$\Rightarrow \boxed{\frac{\lambda e^{t\alpha}}{\lambda - t}}$

(ii) $M_x'(t) = \lambda(e^{ta})(\lambda-t)^{-2}(-1)(-1) + \lambda(\lambda-t)^{-1}(e^{t\alpha})(\alpha)$

$= \frac{\lambda e^{t\alpha}}{(\lambda-t)^{+2}} + \frac{\lambda e^{t\alpha} \alpha}{\lambda - t}$

$E(x) \Rightarrow M_x'(0) = \frac{\lambda(1)}{\lambda^2} + \frac{\lambda \alpha}{\lambda}$

$= \frac{1 + \lambda \alpha}{\lambda} \Rightarrow \boxed{\frac{1+\lambda\alpha}{\lambda}}$

$M_x''(t) \Rightarrow \left[\lambda e^{t\alpha} (-2)(\lambda-t)^{-3}(-1) \right] + \left[\lambda (\lambda - t)^{-2} (e^{t\alpha})(\alpha) \right] +$

$\left[\lambda \alpha e^{t\alpha} (\lambda - t)^{-2} (-1)(-1) \right] + \left[\lambda \alpha (\lambda - t)^{-1} (e^{t\alpha})(\alpha) \right.$

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$$= \frac{2\lambda e^{t\alpha}}{\cancel{0}(\lambda-t)^3} + \frac{\lambda e^{t\alpha}\alpha}{(\lambda-t)^2} + \frac{\lambda\alpha e^{t\alpha}}{(\lambda-t)^2} + \frac{\alpha^2\lambda e^{t\alpha}}{(\lambda-t)}$$

$$= \frac{2\lambda e^{t\alpha}}{(\lambda-t)^3} + \frac{2\lambda e^{t\alpha}\alpha}{(\lambda-t)^2} + \frac{\alpha^2\lambda e^{t\alpha}}{(\lambda-t)}$$

$$E(x^2) \Rightarrow M''_x(0) \Rightarrow \frac{2\cancel{\lambda}(1)}{\lambda^{\cancel{3}}} + \frac{2\cancel{\lambda}\alpha}{\lambda^{\cancel{2}}} + \frac{\cancel{\lambda}\alpha^2}{\cancel{\lambda}}$$

$$\Rightarrow \frac{2}{\lambda} + \frac{2\alpha}{\lambda} + \alpha^2$$

$$v(x) \Rightarrow E(x^2) - [E(x)]^2$$

$$\Rightarrow \alpha^2 + \frac{2\alpha}{\lambda} + \frac{2}{\lambda^2} - \left(\frac{1+\lambda^2\alpha^2+2\alpha\lambda}{\lambda^2}\right)$$

$$\Rightarrow \cancel{\alpha^2} + \cancel{\frac{2\alpha}{\lambda}} + \frac{2}{\lambda^2} - \frac{1}{\lambda^2} - \cancel{\alpha^2} - \cancel{\frac{2\alpha}{\lambda}}$$

$$\Rightarrow \boxed{\frac{1}{\lambda^2}}$$

4. $G_v(t) = \left(\frac{p}{1-qt}\right)^k$, $p+q=1$ negative binomial

$$P(v=4) = \frac{\sqrt{k+4}}{\sqrt{5}\sqrt{k}}\, p^k q^k$$

$$\Rightarrow \frac{\sqrt{k+4}}{4!\sqrt{k}}\, p^k q^k$$

$$\Rightarrow \boxed{\frac{(k+\cancel{0}3)!}{(k-1)! \times 4!}\, p^k q^k}$$

5. $f(x,y) = ax^2 + axy \qquad 0 < x < 5 \quad \text{and} \quad 10 < y < 20$

$$\rightarrow \quad 1 = \int_{10}^{20}\int_0^5 (ax^2 + axy)\, dx \cdot dy$$

$$= \int_{10}^{20} \left[\frac{ax^3}{3}\right]_0^5 + \left[\frac{ax^2 y}{2}\right]_0^5 dy$$

$$\Rightarrow \int_{10}^{20} \frac{a}{3}(125-0) + \frac{ay}{2}[25-0]\, dy$$

$$\Rightarrow \int_{10}^{20} \frac{125a}{3} + \frac{25ay}{2}\, dy$$

$$\Rightarrow \frac{125a}{3}[y]_{10}^{20} + \frac{25a}{2}\left[\frac{y^2}{2}\right]_{10}^{20}$$

$$\Rightarrow \frac{125a(10)}{3} + \frac{25a}{4}(400-100)$$

$$\Rightarrow \frac{1250a}{3} + 1875a$$

$$a \Rightarrow \boxed{\frac{3}{6875}}$$

$$\rightarrow \quad f_X(x) \Rightarrow \int_{10}^{20} (x^2 + xy)\, a \cdot dy$$

$$\Rightarrow \frac{3}{6875} \int_{10}^{20} (x^2 + xy)\, dy$$

$$\Rightarrow \frac{3}{6875}\left[x^2[y]_{10}^{20} + \frac{x}{2}[y^2]_{10}^{20}\right]$$

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$$\Rightarrow \boxed{\frac{3}{6875}\left(10x^2+150x\right)}$$

$$\rightarrow E(Y|X) \Rightarrow y f(X|Y) \Rightarrow y\int_{10}^{20} \frac{f(x,y)}{f_x(x)}$$

$$\Rightarrow \int_{10}^{20} \frac{y(\cancel{x})(x^2+xy)}{\cancel{x}(10x^2+150x)}\,dy$$

$$\rightarrow \int_{10}^{20} \frac{y\cancel{x}(x+y)}{10\cancel{x}(x+15)}\,dy$$

$$\Rightarrow \frac{1}{10(x+15)}\left[\frac{x\cdot y^2}{2}+\frac{y^3}{3}\right]_{10}^{20}$$

$$\frac{1}{10(x+15)}\left[\frac{x}{2}\left(\cancel{300}^{150}\right)+\frac{1}{3}\left(8000-1000\right)\right]$$

$$\frac{1}{1\cancel{0}(x+15)}\left(15\cancel{0}x+\frac{700\cancel{0}}{3}\right) \Rightarrow \boxed{\left(15x+\frac{700}{3}\right)\frac{1}{(x+15)}}$$

6. $X \sim Poin(\lambda)$ $\quad$ $Y \sim Poin(\mu)$

$M_X(t) = e^{\lambda(e^t-1)}$ $\qquad$ $M_Y(t) = e^{\mu(e^{\mu}-1)}$

$M_{X-Y}(t) \Rightarrow E(e^{t(x-y)} \Rightarrow E\, e^{tx-yt}$

$\Rightarrow \frac{E\, e^{xt}}{E(e^{\lambda t}} = \frac{E\, e^{\lambda(e^t-1)}}{e^{\mu(e^t-1)}} = \boxed{e^{(e^t-1)(\lambda-\mu)}}$

$X-Y \sim Poin(\lambda-\mu)$ is proved.

Now,

$M_{X+Y}(t) = E(e^{t(x+y)})$

$\Rightarrow E\, e^{tx} \cdot E\, e^{ty}$

$E\left[e^{\lambda(e^t-1)} \cdot e^{\mu(e^t-1)}\right]$

$\Rightarrow e^{(e^t-1)(\lambda+\mu)}$

$X+Y \sim Poin(\lambda+\mu)$.

7.(i) $Var(X|Y) \Rightarrow E(X^2|Y=2) - (E(X|Y=2))^2$

$\Rightarrow \frac{1^2(0)+2^2(0.3)+3^2(0.1)+4^2(0.2)}{0.6} - \left(\frac{1(0)+2(0.3)+3(0.1)+4(0.2)}{0.6}\right)^2$

$\Rightarrow 8.83333 - (2.83333)^2$

$8.8333 - 8.02777 \Rightarrow \boxed{0.805552224}$

(ii) $f(u,v) = \frac{48}{67}(2uv-u^2)$, $0<u<1$ $\quad$ $\frac{v}{2}<v<2$

$E(u|V=v) = \int_0^1 u\, f(u|V=v)\, du$

$\int_0^1 u \cdot \frac{f(u,V=v)}{f(V=v)}\, du$

$$f_V(v=v) \Rightarrow \int_0^1 \frac{48}{67}(2uv-u^2)\,du$$

$$\Rightarrow \frac{48}{67}\left[\frac{\cancel{2}v.u^2}{\cancel{2}} - \frac{u^3}{3}\right]_0^1 = \frac{48}{67}\left[v(1-0) - \frac{1}{3}(1-0)\right]$$

$$\frac{48}{67}\left[v-\frac{1}{3}\right] \Rightarrow \boxed{\frac{48}{67}\left[\frac{3v-1}{3}\right]}$$

$$E(U|V=v) \Rightarrow \int_0^1 \frac{u\left(\cancel{\frac{48}{67}}\right)(2uv-u^2)}{\cancel{\frac{48}{67}}\left(\frac{3v-1}{3}\right)}\,du$$

$$\Rightarrow \int_0^1 \cancel{(2v}\; u \cdot u(2v-u) \times \left(\frac{3}{3v-1}\right) du$$

$$\Rightarrow \frac{3}{3v-1}\int_0^1 \left[\frac{2 \cdot u^3 v}{3} - \frac{\cancel{u}\,u^4}{\cancel{2}\,4}\right]_0^1$$

$$\frac{3}{3v-1}\left[\frac{2v}{3}(1-0) - \frac{1}{4}(1-0)\right]$$

$$\boxed{\frac{3}{3v-1}\left[\frac{2v}{3}-\frac{1}{4}\right]}$$

8. $X \sim \text{Gamma}(3, 2)$

$E(Y|X=x) = 3x+1$

$E(E(Y|X=x)) = E(Y)$

$E(Y) \Rightarrow E(3X+1)$

$\Rightarrow 3E(x)+1$

$\Rightarrow 3\cancel{E}\left(\frac{3}{2}\right)+1$

$\boxed{\frac{11}{2}}$

$Var(Y|X=x) = 2x^2+5$

$var(var(Y|X=x)) = V(Y)$

$V(Y) \Rightarrow E(Var(Y|X=x))$
$+ V(E(Y|X=x))$

$\Rightarrow \boxed{E(2x^2+5) + V(3x+1)}$

$\Rightarrow 2E(x^2) + 5 + (3)^2 V(x)$

$\Rightarrow 2(3) + 5 + 9\left(\frac{3}{4}\right)$

$\Rightarrow 6 + 5 + \frac{27}{4} \Rightarrow \boxed{17.75}$

9. $Z = X+Y$

$E(x) = 3$

$SD = \sqrt{V(x)} = 2$

$E(y) = 4$

$S.D. = \sqrt{V(\cancel{x}Y)} = 1$

(i) $Cov(X,Z) \Rightarrow Cov(X, X+Y)$

$\Rightarrow Cov(X,X) + Cov(X,Y)$

$\Rightarrow V(X) + (-0.6)$

$\Rightarrow (2)^2 + (-0.6) \Rightarrow \boxed{3.4}$

(ii) $var(Z) \Rightarrow var(X+Y)$

$= V(x) + V(\cancel{z}y) + 2cov(x,y)$

$\Rightarrow (2)^2 + (1)^2 + 2(-0.6)$

$5 + (-1.2) \Rightarrow \boxed{3.8}$

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10. $f(x,y) = k x^{-\alpha} \cdot e^{-y/\beta}$ when $1 < x < \infty$ and $1 < y < \infty$

$\alpha > 1$ & $\beta > 0$

$$1 = k \int_1^\infty x^{-\alpha}\, dx \int_1^\infty e^{-y/\beta}\, dy$$

$$1 = k \left[\frac{x^{-\alpha+1}}{-\alpha+1} \right]_1^\infty \left[\frac{e^{-y/\beta}}{-1/\beta} \right]_1^\infty$$

$$1 = k \left[0 - \frac{1}{(-\alpha+1)} \right] \left[0 + \beta \left(e^{-\frac{1}{\beta}} \right) \right]$$

$$\Rightarrow k \left[\frac{-1}{(-\alpha+1)} \right] \left[\beta (e)^{-\frac{1}{\beta}} \right]$$

$$\boxed{k \Rightarrow \frac{(-\alpha+1)}{1} \left[\frac{1}{\beta (e)^{-\frac{1}{\beta}}} \right]}$$