Probability & Statistics
Assignment - 1

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Q1. $\dfrac{\hat{\lambda}-\lambda}{\sqrt{\lambda/n}} \sim N(0,1)$

$\hat{\lambda} = \dfrac{\sum x_i}{n}$

$\hat{\lambda} = 200$

$P(Z_{2.5\%} < Z < Z_{97.5\%}) = 0.95$

$P(-1.96 < Z < 1.96) = 0.95$

$P\left(-1.96 < \dfrac{\hat{\lambda}-\lambda}{\sqrt{\hat{\lambda}/n}} < 1.96\right) = 0.95$

$P\left(-1.96\sqrt{\dfrac{\hat{\lambda}}{n}} - \hat{\lambda} < \lambda < 1.96\sqrt{\dfrac{\hat{\lambda}}{n}} - \hat{\lambda}\right) = 0.95$

$P\left(\hat{\lambda} - 1.96\sqrt{\dfrac{\hat{\lambda}}{n}} < \lambda < \hat{\lambda} + 1.96\sqrt{\dfrac{\hat{\lambda}}{n}}\right) = 0.95$

95% confidence interval of λ when $\hat{\lambda} = 200$ and $n = 1$,

95% CI for $\lambda = \left(200 - 1.96\sqrt{200},\ 200 + 1.96\sqrt{200}\right)$

$= (172.28141, 227.7185)$

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Q2. X_i, $i = 1, 2 \ldots, n$ are i.i.d.

Mean $= \mu$

Variance $= \sigma^2$

$$\bar{X} = \frac{\Sigma X_i}{n}$$

$$S^2 = \frac{1}{n-1}\left(\Sigma X_i^2 - n\bar{X}^2\right)$$

(i) $$E(\bar{X}) = E\left(\frac{\Sigma X_i}{n}\right)$$

$$= \frac{\Sigma E(X_i)}{n}$$

$$= \frac{\cancel{n}\mu}{\cancel{n}}$$

$$= \mu$$

$$V(\bar{X}) = V\left(\frac{\Sigma X_i}{n}\right)$$

$$\cancel{= \frac{\Sigma V(X_i)}{n}}$$

$$= \frac{1}{n^2} V(\Sigma X_i)$$

$$= \frac{1}{n^2} \Sigma V(X_i)$$

$$= \frac{\cancel{n}\sigma^2}{\cancel{n^2}n}$$

$$= \frac{\sigma^2}{n}$$

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(ii) To show $E(S^2) = \sigma^2$

$$E(S^2) = E\left[\frac{1}{n-1}(\Sigma x_i^2 - n\bar{x}^2)\right]$$

$$= \frac{1}{n-1}\left[E(\Sigma x_i^2) - nE(\bar{x}^2)\right]$$

$$= \frac{1}{n-1}\left[\Sigma E(x_i^2) - nE(\bar{x}^2)\right]$$

Now,

$$\Rightarrow V(X_i) = E(x_i^2) - [E(x)]^2$$

$$E(x_i^2) = \mu^2 + \sigma^2$$

$$V(\bar{X}) = E(\bar{x}^2) - [E(\bar{x})]^2$$

$$E(\bar{X}^2) = \mu^2 + \frac{\sigma^2}{n}$$

$$E(S^2) = \frac{1}{n-1}\left[\Sigma(\mu^2+\sigma^2) - n\left(\mu^2+\frac{\sigma^2}{n}\right)\right]$$

$$= \frac{1}{n-1}\left(\cancel{n\mu^2} + n\sigma^2 - \cancel{n\mu^2} - \sigma^2\right)$$

$$= \frac{1}{\cancel{n-1}} \times \sigma^2 \cancel{(n-1)}$$

$$E(S^2) = \sigma^2$$

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(iii) Now, $X \sim N(\mu, \sigma^2)$

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$V\left(\frac{(n-1)S^2}{\sigma^2}\right) = \frac{(n-1)^2}{\sigma^4} V(S^2)$$

$$V\left(\chi^2_{n-1}\right) = \frac{(n-1)^2}{\sigma^4} V(S^2)$$

$$2\cancel{(n-1)} = \frac{\cancel{(n-1)^2}^{(n-1)}}{\sigma^4} V(S^2)$$

$$V(S^2) = \frac{2\sigma^4}{n-1}$$

(iv) We need,

~~$P(S^2 = (0.5)S^2$~~

$$P(0.5\sigma^2 < S^2 < 1.5\sigma^2)$$
$$= P(S^2 < 1.5\sigma^2) - P(S^2 < 0.5\sigma^2)$$

$n = 10$
$\sigma^2 = 100$

$$P(S^2 < 1.5\sigma^2) = P\left(S^2 < 150\right)$$
$$= P\left(\frac{9S^2}{\sigma^2} < \frac{150 \times 9}{100}\right)$$
$$= P\left(\chi^2_9 < 13.5\right)$$
$$= 0.8587$$

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$$P(S^2 < 0.5S^2) = P(S^2 < S_0)$$
$$= P\left(\frac{9S^2}{\sigma^2} < \frac{50 \times 9}{100}\right)$$
$$= P\left(\chi^2_9 < 4.5\right)$$
$$= 0.1245$$

$$P(50 < S^2 < 150) = 0.8587 - 0.1245$$
$$= 0.7342$$

Q3. Let no. of claims be a random variable X.

$$X \sim Bin(4, p)$$

(i) $$L = \prod_{i=1}^{180} f(x_i)$$
$$= (P(X=0))^{86} \times (P(X=1))^{75} \times [P(X=2)]^{16} \times [P(X=3)]^2 \times [P(X=4)]^1$$

$$= \left({}^4C_0 p^0 (1-p)^4\right)^{86} \times \left({}^4C_1 p^1 (1-p)^3\right)^{75} \times \left({}^4C_2 p^2 (1-p)^2\right)^{16} \times \left({}^4C_3 p^3 (1-p)\right)^2 \times \left({}^4C_4 p^4\right)^1$$

$$= \text{constant} \times p^{117} \times (1-p)^{603}$$

$$\log L = \log(\text{constant}) + 117 \log(p) + 603 \log(1-p)$$

$$\frac{dl}{dp} = 0 + \frac{117}{p} - \frac{603}{(1-p)}$$

$$\frac{dl}{dp} = \frac{117}{p} - \frac{603}{(1-p)}$$

$$\frac{dl}{dp} = 0$$

$$\frac{117}{p} = \frac{603}{1-p}$$

$$117 - 117p = 603p$$

$$720p = 117$$

$$\hat{p} = \frac{117}{720}$$

$$\hat{p} = 0.1625$$

(ii) Goodness of fit test

H_0 : no. of claims conforms to the binomial model
H_1 : no of claims does not conform to the binomial model

	0	1	2	3	4
O_i	86	75	16	2	1
E_i	88.542	68.724	19.998	2.574	0.108

$p = 0.1625$

$P(X=0) = {}^4C_0\, p^0 (1-p)^4 = 0.4919$
$P(X=1) = {}^4C_1\, p^1 (1-p)^3 = 0.3818$
$P(X=2) = 0.1111$
$P(X=3) = 0.0143$
$P(X=4) = 0.0006$

O_i	86	75	19
E_i	88.542	68.724	22.68

$$T.S = \sum \frac{(O_i - E_i)^2}{E_i} \sim \chi^2_{5-1-1-2}$$

$$1.24322 \sim \chi^2_1$$

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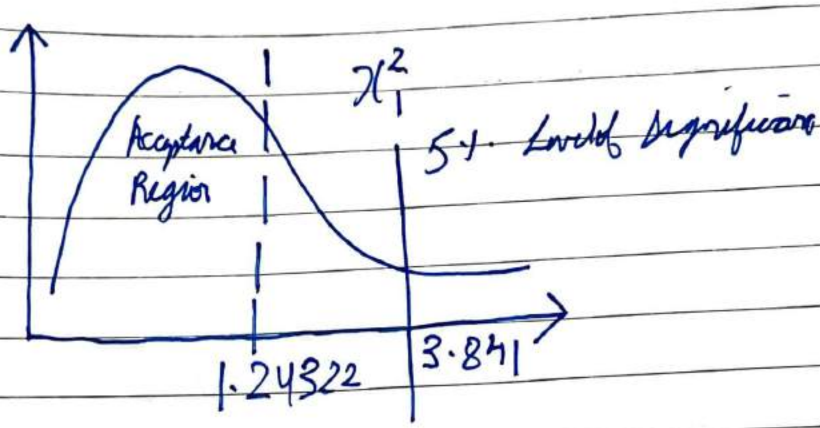

We have insufficient evidence to reject H_0 at 5% Level of Significance. Therefore, no. of claims conforms to the binomial model.

Q4. $f(x) = \dfrac{c\beta^3}{(x+\beta)^4}$; $x>0, \beta>0$

(i) Comparing the above $f(x)$ to ~~f(x)~~ P.D.F of Pareto distribution (2 parameter version)

For pareto,

$$PDF = \frac{\alpha\lambda^\alpha}{(\lambda+x)^{\alpha+1}}$$

Thus, β corresponds to α.

$\Rightarrow$ Thus, $c = 3$. (as c corresponds to α here)

$\Rightarrow E(X) = \dfrac{\lambda}{\alpha-1}$

$E(X) = \dfrac{\beta}{2}$

$\Rightarrow V(X) = \dfrac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} = \dfrac{3\beta^2}{4}$

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(i) $E(X) = \frac{\beta}{2}$

$E(X) = \bar{X}$

$\frac{\beta}{2} = \bar{X}$

$\hat{\beta} = 2\bar{X}$

$\Rightarrow$ Bias $= E(\hat{\beta}) - \beta$

$= E(2\bar{X}) - \beta$

$= 2E(\bar{X}) - \beta$

$= \frac{2}{n}\Sigma E(X_i) - \beta$

$= \frac{\cancel{2}}{\cancel{n}} \times \frac{\cancel{n}\beta}{\cancel{2}} - \beta$

$= \beta - \beta$

Bias $= 0$, hence unbiased.

$\Rightarrow$ MSE $= V(\hat{\beta}) + (\text{Bias})^2$

$= V(\hat{\beta})$

$= V(2\bar{X})$

$= 4V(\bar{X})$

$= 4V\left(\frac{\Sigma X_i}{n}\right)$

$= \frac{4}{n^2}\Sigma V(X_i)$

$= \frac{\cancel{4}}{\cancel{n^2}\, n} \times \frac{3\beta^2 \cancel{n}}{\cancel{4}}$

MSE $= \frac{3\beta^2}{n}$

As n increases, MSE tends to 0.
Hence, $\hat{\beta}$ is consistent.

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(iii) $\hat{\beta} = b\bar{X}$

$$E(\hat{\beta}_2) = E(b\bar{X})$$
$$= bE\left(\frac{\Sigma x_i}{n}\right)$$
$$= \frac{b}{n}\Sigma E(X_i)$$
$$= \frac{b}{\cancel{n}} \times \frac{\beta}{2} \times \cancel{n}$$
$$= \frac{b\beta}{2}$$

$$V(\hat{\beta}_2) = V(b\bar{X})$$
$$= b^2 V\left(\frac{\Sigma x_i}{n}\right)$$
$$= \frac{b^2}{n^2}\Sigma V(X_i)$$
$$= \frac{b^2}{\cancel{n^2}\,n} \times \frac{3}{4}\beta^2 \times \cancel{n}$$
$$= \frac{3}{4}\frac{b^2\beta^2}{n}$$

$$MSE = V(\hat{\beta}_2) + (Bias)^2$$
$$= \frac{3b^2\beta^2}{4n} + \left(\frac{b\beta}{2} - \beta\right)^2$$
$$= \frac{3b^2\beta^2}{4n} + \left(\frac{b\beta - 2\beta}{2}\right)^2$$
$$= \frac{3b^2\beta^2}{4n} + \left(\frac{b^2\beta^2 + 4\beta^2 - 4\beta^2 b}{4}\right)$$
$$= \frac{3b^2\beta^2 + nb^2\beta^2 + 4n\beta^2 - 4n\beta^2 b}{4n}$$
$$M = \frac{\beta^2}{4n}(3b^2 + nb^2 + 4n - 4nb)$$

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$$\frac{dM}{db} = \frac{\beta^2}{4n}(6b + 2nb - 4n)$$

$$\frac{dM}{db} = 0$$

$$\frac{\beta^2}{4n}(6b+2nb-4n) = 0$$

$$6b+2nb-4n = 0$$

$$3b+nb-2n = 0$$

$$b(3+n) = 2n$$

$$b = \frac{2n}{3+n}$$

$$b = \frac{2}{\left(\frac{3+n}{n}\right)}$$

$$b = \frac{2}{\left(1+\frac{3}{n}\right)}$$

$$\frac{d^2M}{db^2} = \frac{\beta^2}{4n}(6+2n)$$

$$\frac{\beta^2}{4n}(6+2n) > 0$$

Hence, $\frac{d^2M}{db^2}$ is minimum.

Thus, $b = \frac{2}{(1+3/n)}$ minimises the MSE.

iv) $E(\hat{\beta}_2) = \dfrac{b\beta}{2}$

$= \dfrac{\cancel{2}}{(1+3/n)} \times \dfrac{\beta}{\cancel{2}}$

$= \dfrac{n\beta}{(n+3)}$

Bias $= E(\hat{\beta}_2) - \beta$

$= \dfrac{n\beta - \beta(n+3)}{n+3}$

$= \dfrac{\cancel{n\beta} - \cancel{n\beta} - 3\beta}{n+3}$

$= -\dfrac{3\beta}{n+3}$

Thus, we have a negative bias & hence it underestimates the true parameter β.

$\cancel{V(\hat{\beta}_2)}$ MSE $= \dfrac{\beta^2}{4n}\left(3b^2 + nb^2 + 4n - 4nb\right)$

$= \dfrac{\beta^2}{4n}\left[b^2(n+3) + 4n(1-b)\right]$

$= \dfrac{\beta^2}{4n}\left[\dfrac{4n^2 \times \cancel{(n+3)}}{\cancel{(n+3)^2}(n+3)} + 4n\left(\dfrac{3+n-2n}{n+3}\right)\right]$

$= \beta^2\left[\dfrac{4n^2}{n+3} + 4n\left(\dfrac{3-n}{n+3}\right)\right]$

$= \beta^2\left[\dfrac{\cancel{4n^2} + 12n - \cancel{4n^2}}{n+3}\right]$

MSE $\cancel{= \dfrac{12\beta^2 n}{n+3}} = \dfrac{12n\beta^2}{n+3}$

As $n \to \infty$, M.S.E doesnot tend to 0.
Hence, it is inconsistent.

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Q5. (i) $X \sim U(a,b)$

$\rightarrow E(X) = \frac{a+b}{2}$

$\bar{X} = \frac{a+b}{2}$

& $a = 2\bar{X} - b$ — ①

$\rightarrow V(X) = \frac{(b-a)^2}{12}$

$S^2 = \frac{(b-a)^2}{12}$

$12S^2 = (b-a)^2$

$2\sqrt{3}S = b-a$

$2\sqrt{3}S = 2\bar{x} - a - a \quad \{b = 2\bar{x} - a\}$

$2\sqrt{3}S = 2\bar{x} - 2a$

$\sqrt{3}S = \bar{X} - a$

$\hat{a} = \bar{X} - \sqrt{3}S$

(ii) $b = 2\bar{X} - a$

$b = 2\bar{X} - \bar{X} + \sqrt{3}S$

$\hat{b} = \bar{X} + \sqrt{3}S$

(iii) Sample of observat^ns: 1, 2, 8, 50, 150

$\bar{X} = 42.2$

$S = 63.57$

$\hat{a} = \bar{X} - \sqrt{3}S$

$= -67.906$

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$$\hat{b} = \bar{X} + \sqrt{3}S$$
$$= 152.3064$$

Hence we can conclude that value of $\hat{b}$ is approx very close to sample value but $\hat{a}$ is very far from the true sample value, so it is the potential weakness of method of moments

(iv) $X \sim U(0, b)$

$$L = \prod_{i=1}^{n} f(x_i)$$
$$= \left(\frac{1}{b}\right)^n$$

$$\log L = -n \log b$$
$$\frac{dl}{db} = -\frac{n}{b}$$

$\frac{dl}{db} = 0$, gives $b = \infty$ which is not possible.

Also,
$$\frac{d^2 l}{db^2} = \frac{n}{b^2}$$

$\frac{n}{b^2} > 0$, which we have a minimum likelihood estimate

Thus, we say, $\hat{b} = \max(X_i)$

Q6. $H_0 : p = 0.1$

$H_1 : p > 0.1$

(i.) $P(\text{Type I error}) = P(\text{rejecting } H_0 \mid H_0 \text{ is true})$

Let X be the no. of hits in first 12 missile and Y be the no. of hits in step 12 missiles.

$$P(\text{Type I error}) = P(X \geq 3 \mid p=0.1) + P(X=0 \mid p=0.1)\, P(Y \geq 5 \mid p=0.1) + P(X=1 \mid p=0.1)\, P(Y \geq 4 \mid p=0.1) + P(X=2 \mid p=0.1)\, P(Y \geq 3 \mid p=0.1)$$

$$= (1-0.8891) + 0.2824(1-0.9957) + (0.6590-0.2824)(1-0.9744) + (0.8891-0.6590)(1-0.8891)$$

$$= 0.14727 \approx 15\%$$

~~15%~~

(ii) Probability of rejecting null hypothesis when $p = 0.3$.

$$= P(X \geq 3 \mid p=0.3) + P(X=0 \mid p=0.3)\, P(X \geq 5 \mid p=0.3) + P(X=1 \mid p=0.3)\, P(X \geq 4 \mid p=0.3) + P(X=2 \mid p=0.3)\, P(Y \geq 3 \mid p=0.3)$$

$$= (1-0.2528) + 0.0138(1-0.7237) + (0.0850-0.0138)(1-0.4925) + (0.2528-0.0850)(1-0.2528)$$

$$= 0.9125 \approx 91\%$$

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(iii) Probability of type II error
Accepting H_0 when H_0 is false

$$= 1 - 0.91253$$
$$= 0.08747$$
$$\approx 9\% \text{ (where } p = 0.3)$$

(iv) 10 space agencies in aggregate fired 120 missiles and recorded 40 hits.

$$\frac{X/n - p}{\sqrt{\frac{p(1-p)}{n}}} \sim N(0,1)$$

$n = 120$, $X = 40$
$$\hat{p} = \frac{40}{120} = \frac{1}{3} = 0.3333$$
$$Z_{10\%} = 1.2816$$

Lower bound of 90% right tailed confidence interval for p is,

$$\hat{p} - 1.2816\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.3333 - 1.2816\sqrt{\frac{0.3333(1-0.3333)}{120}}$$
$$= 0.2782$$

(v) $p > 0.278$ at 10% LOS.
$H_0: p = 0.278$ vs $H_1: p > 0.278$
One sample binomial model
$$\frac{X - np_0}{\sqrt{np_0q_0}} \sim N(0,1) \text{ with continuity correct}^n$$

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$$\frac{39.5 - 120(0.278)}{\sqrt{120 \times 0.278 \times 0.722}} = 1.2511$$

We don't have sufficient evidence to reject H_0 at 10% level of significance

(vi) Lower bound of confidence interval implies that there is only a 10% chance of $p \le 0.2782$ whereas from the hypothesis test, ~~...~~ 'p' could be less than or equal to 0.2780. with probability of more than 10%.

(vii) $H_0: \mu_1 = \mu_2$ vs $H_1: \mu_1 \neq \mu_2$

$$T.S = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \sim t_{n_1+n_2-2}$$

Given,

$\bar{X}_1 = 65$

$\bar{X}_2 = 70$

$S_1 = 54$

$S_2 = 70$

$n_1 = 12$

$n_2 = 15$

$$S_p^2 = \frac{1}{25}\left(11(2916) + 14(4900)\right) = 4027.04$$

$$\frac{65-70}{65.451\sqrt{\frac{1}{12} + \frac{1}{15}}} = -0.2034$$

There is $\pm$ 2.060 ($= t_{25,\, 2.5\%}$)

So we have insufficient evidence to reject H_0 at 5% level.

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Q7 (i) Public   Private

$\bar{X}_1 = 30$ $\bar{X}_2 = 35.0555$

$S_1^2 = 64.3125$ $S_2^2 = 67.40287$.

(ii) 2 sided t-test can be applied in case the samples come from populations with equal variance.

We are testing $H_0: \sigma_1^2 = \sigma_2^2$ vs $H_1: \sigma_1^2 \neq \sigma_2^2$

Test statistic is $\frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} \sim F_{n_1-1, n_2-1}$

Value of statistic is $\frac{64.31}{67.42} = 0.9542$

$F_{8,8}$ values at 5% levels are 0.2256 & 4.433. Since the value of the test statistic b/w the above values, we have insufficient evidence to reject the hypothesis and conclude that the population variances are equal.

~~(iii) $H_0: \mu_{Pr} = \mu_{PV}$~~

~~$H_1: \mu_{PV} >$~~

(iii) $H_0: \mu_{Pr} = \mu_{Pu}$

$H_1: \mu_{Pr} > \mu_{Pu}$

$T.S = \frac{(\bar{X}_2 - \bar{X}_1) - (\mu_2 - \mu_1)}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$

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$$= \frac{35.0555 - 30}{8.1152\sqrt{\frac{2}{9}}}$$

$T.S = 1.32151$

Tabulated value $= t_{16, 5\%}$

$= 1.746$

Since, $T.S_{cal} <$ Tabulated value,

We have insufficient evidence to reject H_0.

Q8.

(i) Unbiased estimator

→ An unbiased estimator is an estimator whose expected value is equal to the parameter.
→ Estimator is said to be unbiased if bias is equal to zero for all values of parameter θ.

(ii) Biased

→ A biased estimator is one which delivers an estimate which is consistently different from the parameter to be estimated

(iii) The mean square error of an estimator $\hat{\theta}$ is defined by,
$MSE(\hat{\theta}) = var(\hat{\theta}) + Bias^2$

(iv) The estimator having less MSE is more consistent.

(v) They both would be considered to be consistent when MSE minimises such that when, $n \to \infty$, $MSE \to 0$.

(vi) (i) Method of moments.
(ii) Maximum likelihood estimate.

Q9.

(i) Variable t_K is defined as $t_K \Rightarrow \frac{N(0,1)}{\sqrt{\chi^2_K / K}}$

where K denotes the degree of freedom
and the 2 random variables $N(0,1)$ and χ^2_K are independent.

~~(ii) Mean for $t_K = 0$, for~~

(ii) For $k > 2$

$$E(t_K) = 0$$

$$V(t_K) = \frac{k}{k-2}$$

(iii) (a) $\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{n-1}$

$$P\left(t_{9, 0.5\%} < \frac{\bar{X} - \mu}{S/\sqrt{n}} < t_{9, 99.5\%}\right) = 0.99$$

$$P\left(\bar{X} - t_{9, 0.5\%} \frac{S}{\sqrt{n}} < \mu < \bar{X} + t_{9, 0.5\%} \frac{S}{\sqrt{n}}\right) = 0.99$$

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$t_{9, 0.5\%} = 3.250$

99% Confidence interval for $\mu = \left(50 - 3.25 \times \frac{\sqrt{48.667}}{\sqrt{10}}, 50 + 3.25 \times \frac{\sqrt{48.667}}{\sqrt{10}}\right)$

$= (42.83, 57.17)$

(b). From (ii),

$$t_{n-1} \sim N\left(0, \sqrt{\frac{n-1}{n-3}}\right)$$

$$t_{n-1} \sim N\left(0, \sqrt{\frac{9}{7}}\right)$$

i.e

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim N\left(0, \sqrt{\frac{9}{7}}\right)$$

$$\frac{\bar{X} - \mu}{S/\sqrt{7n/9}} \sim N(0, 1)$$

Critical values level of confidence 2.58.

Confidence interval $= \left(50 - 2.58\sqrt{\frac{48.667 \times 9}{10 \cdot 7}}, 50 + 2.58\sqrt{\frac{48.667 \times 9}{10 \cdot 7}}\right)$

Confidence interval $= (43.54, 56.45)$.

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Q10. $n = 1000$

$\lambda = 3$

$X \sim Poi(n\lambda)$

$X \sim Poi(3000)$

(i) $P(\text{Type I error}) = P(\text{reject } H_0 \mid H_0 \text{ is true})$

$= P(X > \cancel{3000}\ 3100 \mid X \sim Poi(3000))$

Using normal approx.

$P(X > 3100) = P\left(Z > \dfrac{3100 - 3000}{\sqrt{3000}}\right)$

$= 1 - P(Z < 1.825)$

$= 3.39\%$

(ii) Type II error = event of accepting the hypothesis when it is false

(iii) Power of test = $P(\text{rejecting } H_0 \mid H_0 \text{ is false})$

$= P[X > 3100 \mid X \sim Poi(n\lambda) - N(n\lambda, n\lambda)]$

$= 1 - P\left(Z < \dfrac{3100 - n\lambda}{\sqrt{n\lambda}}\right)$

The value of power of test will depend on the value of parameter under alternative hypothesis.

(iv) $\hat{\mu}$ is estimator of μ, $\hat{\mu}$ follows $N\left(\mu, \dfrac{\hat{\mu}}{n}\right)$

Hence, $\dfrac{\hat{\mu} - \mu}{\sqrt{\hat{\mu}/n}} \sim N(0,1)$

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or, $P\left(-2.5758 < \frac{2.9-\mu}{\sqrt{2.9/1000}} < 2.5758\right) = 0.98$

Hence, the confidence interval for μ is $(2.7613, 3.0387)$.

Q11. $n = 12$
$\Sigma X = 213,200$
$\Sigma X^2 = 4,919,860,000$
$\bar{X} = 17,767$
$S = 10,144.$

New Sample mean

$\Sigma X' = \Sigma X - 11,000 - 48,000 + 27,500 + 31,500$
$= 2,13,200$

$\bar{X}' = \frac{2,13,200}{12}$
$= 17,767.$

New Sample standard deviation

$\Sigma X^{2\prime} = 4,919,860,000 - (11,000)^2 - (48,000)^2 + (27500)^2 + (31,500)^2$
$= 4,243,360,000$

$S'^2 = \frac{1}{n-1}\left(\Sigma X^{2\prime} - n(\bar{X}')^2\right)$
$= \frac{1}{11}\left(4,243,360,000 - 12(17,767)^2\right)$
$= 4,1396,775.64$

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$\therefore S' = 6434.032611$

The sample mean remains the same after the inclusion of the 2 permanent employees as the 2 values removed were outliers and the 2 values included ~~maybe around~~ were around the sample mean.

The sample standard deviation reduces as the 2 values now included do not deviate very much from the sample mean causing the variance to reduce and hence the reduction of the sample standard deviation.

Q12

(i) The Cramér-Rao Lower Bound result holds under very general conditions except where the range of the distribution involves the parameter, such as the uniform distribution in this case.
This is due to a discontinuity, so the derivative in the formula does not make sense.

(ii) We have $Y = \max X_i$
Since each X_i lies b/w 0 and θ, the support for Y will be 0 and θ.

For $0 < y < \theta$.

$$P[Y<y] = P[\max X_i < y]$$
$$= P\left[\bigcap_{i=1}^{n} (X_i<y)\right]$$
$$= \prod_{i=1}^{n} P(X_i<y) \quad [\because X_i\text{'s are independent}]$$

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$$= \prod_{i=1}^{n} \left[\int_0^y \frac{dx}{\theta} \right]$$

$$= \left(\frac{y}{\theta}\right)^n$$

Thus the probability density function of Y will be,

$$g_Y(y) = \frac{d}{dy} P(Y<y) = \frac{n}{\theta^n} \cdot y^{n-1} \text{ for } 0<y<\theta.$$

Now, for any non-negative real no. k,

$$E[Y^k] = \int_0^\theta y^k \frac{n}{\theta^n} \cdot y^{n-1}\, dy$$

$$= \frac{n}{n+k} \int_0^\theta \frac{n+k}{\theta^n} \cdot y^{n+k-1}\, dy$$

$$= \frac{n}{n+k} \cdot \frac{\theta^{n+k} - 0}{\theta^n}$$

$$= \frac{n\theta^k}{n+k}$$

(iii) Bias of estimator $\hat{\theta}(c)$ is given as:

$$\text{Bias}\,(\hat{\theta}(c)) = E(\hat{\theta}(c) - \theta]$$

$$= E[cY - \theta]$$

$$= c.E(Y] - \theta$$

$$= c.\frac{n\theta}{n+1} - \theta$$ (using the results in (b) with $k=1$)

$$= \left(\frac{cn}{n+1} - 1\right)\theta$$

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Mean Square error of the estimator $\hat{\theta}(c)$ is given as:

$$MSE[\hat{\theta}(c)] = E[\{\hat{\theta}(c) - \theta\}^2]$$
$$= E[\{cY - \theta\}^2]$$
$$= E[c^2Y^2 - 2c\theta Y + \theta^2]$$
$$= c^2 E(Y^2) - 2c\theta E(Y) + \theta^2$$
$$= c^2 \frac{n\theta^2}{n+2} - c \cdot \frac{2n\theta^2}{n+1} + \theta^2$$ (using (iii) with $k=1$ & $k=2$]

(iv) For $\hat{\theta}(c)$ to be an unbiased estimator of θ, we need:
Bias$(\hat{\theta}(c_u)) = 0$.

$$\left(\frac{c_u n}{n+1} - 1\right) \cdot \theta = 0$$
$$\Rightarrow c_u = \frac{n+1}{n}$$

(v) In order to minimise $MSE[\hat{\theta}(c)]$, we need

$$0 = \frac{d}{dc} MSE[\hat{\theta}(c)]$$
$$= \frac{d}{dc}\left[c^2 \cdot \frac{n\theta^2}{n+2} - c\frac{2n\theta^2}{n+1} + \theta^2\right]$$
$$= 2c\frac{n\theta^2}{n+2} - \frac{2n\theta^2}{n+1}$$

Thus: $c_m = \frac{2n\theta^2}{n+1} \cdot \frac{n+2}{2n\theta^2} = \frac{n+2}{n+1}$

Note: $\frac{d^2}{dc^2} MSE[\hat{\theta}(c)] = \frac{d}{dc}\left[2c\frac{n\theta^2}{n+2} - \frac{2n\theta^2}{n+1}\right] = \frac{2n\theta^2}{n+2} > 0$ $\Downarrow$ minimum

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(vi) In order to minimise error in estimation, it is preferred to opt for an estimator which has lower mean square error among different competing estimators. So here $\hat{\theta}(c_m)$ will be preferred over $\hat{\theta}(c_\mu)$.

As n becomes large, ~~$\hat{\theta}(c_\mu)$~~ $\hat{\theta}(c_\mu) = \frac{n+1}{n} Y \to Y$.

Similarly $\hat{\theta}(c_m) = \frac{n+2}{n+1} Y \to Y$

Thus the two estimators becomes one and same as $n \to \infty$.